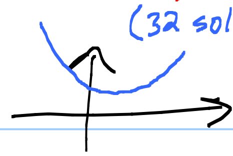
Lesson 32 on 11.9 Complex Fourier Transform

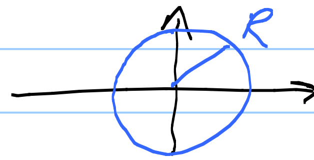
29,30,31 due Wed
32 not to be turned in
(32 sol'ns on Piazza) skip DFT, FFT

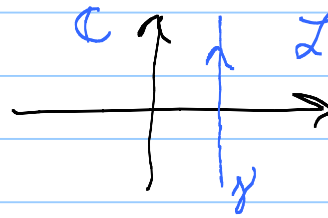
Webex Tues 8-9pm

Going complex: $ax^2+bx+c=0$  always has roots in $\mathbb{C}$.

Fund. Thm. Alg. $a_n x^n + \cdots + a_1 x + a_0 = 0$ has n roots in $\mathbb{C}$. Can factor:
$A(z-r_1)^{m_1}\cdots(z-r_k)^{m_k} = 0 \qquad \sum m_j = n$

Good reasons to do calculus over $\mathbb{C}$:

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$  Radius of conv $= \infty$

$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\,dt$  $\mathcal{L}^{-1} = \int_\gamma \cdots$

let $s = z$, a complex number

Fact: e^{inx} are $\perp$ on $[-\pi,\pi]$: $\int_{-\pi}^{\pi} e^{inx}\left(\overline{e^{imx}}\right)dx$

$$= \int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx$$

$$= \left[\frac{1}{i(n-m)} e^{i(n-m)x}\right]_{x=-\pi}^{\pi} = 0$$

if $n \neq m$

because Cos and Sin are 2π periodic

What if $n=m$?

$$= \int_{-\pi}^{\pi} \underbrace{e^{i\cdot 0}}_{1}\,dx = 2\pi$$

Next step: Let $L \to \infty$ in Complex Fourier Series. Get Complex Fourier Transform (the Fourier Transform).

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\,dx$$

fcn on $\mathbb{R}$ ($\hat{f}(w)$) fcn on $\mathbb{R}$ ($f(x)$)

Need $\int_{-\infty}^{\infty} |f|\,dx < \infty$.

Big fact: $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

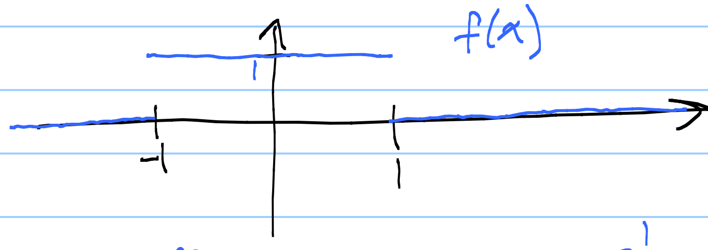
Remark: get midpoints at jumps of f.

Inverse Fourier Transform: Notation $\mathcal{F}[f] = \hat{f}$

$$\mathcal{F}^{-1}[g] = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w)\, e^{iwx}\, dw.$$

Big fact: $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$.

EX:



$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(-iw)} e^{-iwx} \right]_{x=-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-iw} - e^{iw}}{-iw} = \frac{1}{\sqrt{2\pi}} \frac{1}{w}\, 2 \underbrace{\left(\frac{e^{iw} - e^{-iw}}{2i} \right)}_{\sin w}$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin w}{w}$$

Fact: $\|f\|^2 = \|\hat{f}\|^2$ Plancherel Identity

$$\int_{-\infty}^{\infty} |f|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}|\, dw$$

EX: $$\int_{-1}^{1} 1^2\, dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\, \frac{\sin w}{w} \right)^2 dw$$

$$2 = \frac{2}{\pi}\underbrace{\int_{-\infty}^{\infty}\frac{\sin^2 w}{w^2}\,dw}_{I=\pi}$$

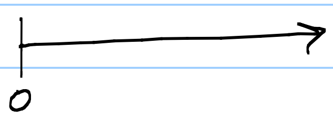
Properties: $\tilde{\mathcal{F}}$ is linear

$$\tilde{\mathcal{F}}[f'] = iw\,\tilde{\mathcal{F}}[f]$$

$$\tilde{\mathcal{F}}[f''] = -w^2\,\tilde{\mathcal{F}}[f]$$

$i^2=-1$

Hot wire:



$u_{xx} = u_t$

Iced end: $u(0,t)=0$ for all t.

Use Fourier Sine Tranform

$$\tilde{\mathcal{F}}_s[u_{xx}] = -w^2\,\tilde{\mathcal{F}}_s[u] + \sqrt{\frac{2}{\pi}}\,w\,\underbrace{u(0,t)}_{=0}$$

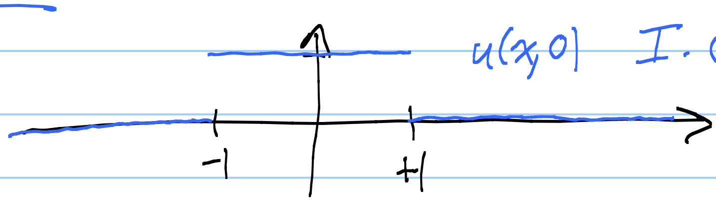
Insulated end: $\frac{\partial u}{\partial x}(0,t)=0$ for all t

temp gradient $=0$
no heat flow

Use Fourier Cosine Trans:

$$\tilde{\mathcal{F}}_c[u_{xx}] = -w^2\,\tilde{\mathcal{F}}_c[u] - \sqrt{\frac{2}{\pi}}\,\underbrace{\frac{\partial u}{\partial x}(0,t)}_{0}$$

No end: Use Fourier Transform.



$u(x,0)$ I.C.

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\,e^{-iwx}\,dx$$

$$\underbrace{\mathcal{F}[u_{xx}]}_{-w^2\hat{u}} = \mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial \hat{u}}{\partial t}$$

diff under $\int$

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -w^2\hat{u}}$$

exp decay in t

w is a param.

$$\hat{u}(w,t) = C(w)e^{-w^2 t}$$

C might depend on w

Aha! $\hat{u}(w,0) = C(w)e^0 = C(w) = \mathcal{F}[I.C] = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}$

Big moment: Undo $\mathcal{F}$ with $\mathcal{F}^{-1}$ to get u!

$$u(x,t) = \mathcal{F}^{-1}\left[\underbrace{\sqrt{\tfrac{2}{\pi}}\,\frac{\sin w}{w}}_{C(w)}\,e^{-w^2 t}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}\,e^{-w^2 t}\,e^{iwx}\,dw$$

even even $\cos wx + i\sin wx$

even odd

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin w\cos wx}{w}\,e^{-w^2 t}\,dw$$

Convolutions:

$$\int_0^t f(\tau)g(t-\tau)\,d\tau = f*g$$

Laplace *

$$\mathcal{L}[f*g] = F(s)\,G(s)$$

$$(f*g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)\,d\tau$$

Fourier *

Fact: $\mathcal{F}[f*g] = \sqrt{2\pi}\,\hat{f}(w)\,\hat{g}(w)$