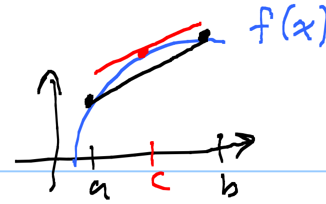
Lesson 36 on 12.1, 12.2 PDE

36, 37, 38 due Wed., Dec. 2
No WebEX next week

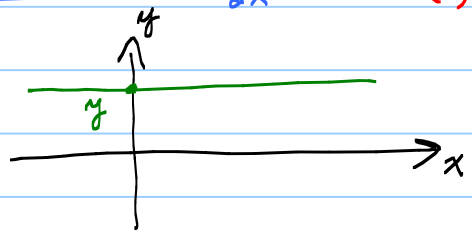


Freshman calculus: Mean Value Theorem $\dfrac{f(b)-f(a)}{b-a} = f'(c)$

MVT $\Rightarrow$ If $f' \equiv 0$, then f is const.

Easiest PDE: $\dfrac{\partial u}{\partial x} \equiv 0$ (*)

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h}$$



PDE $\Rightarrow$ $u(x,y)$ is constant on horiz lines.

$u(x,y) = C(y)$ $\leftarrow$ const. might depend on y

Conversely, If $u(x,y) = g(y)$ $\leftarrow$ fcn of y alone

Then $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x} g(y) = 0$

Fact: Sol^n of (*) is $u(x,y) = C(y)$
where $C(y)$ is an arbitrary fcn of y.

Prob: Solve $\dfrac{\partial u}{\partial x} = e^y \sin 2x$

$$u = \int e^y \sin 2x \, dx = e^y\left(-\tfrac{1}{2}\cos 2x\right) + C(y)$$

treat y like a const.

Why it is right: $\dfrac{\partial}{\partial x}\Big[\underbrace{u - \left(-\tfrac{1}{2}e^y \cos 2x\right)}_{\text{must } = C(y)}\Big] \equiv 0$

EX: $u_{xx} + u = 0$ $\leftarrow$ think of y as a parameter

$$u(x,y) = A(y)\cos x + B(y)\sin x$$

where $A(y)$ and $B(y)$ are arb fcns of y.

First order linear ODE

$$A(x)\frac{dy}{dx} + B(x)y = C(x)$$

Step 1: Divide by $A(x)$:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Standard Form

Step 2: Find integrating factor

$$\mu = e^{\int P(x)dx} \leftarrow \text{no } +C \text{ here}$$

Step 3: Mult. Standform Eqn by μ:

$$\underbrace{\mu\left[\frac{dy}{dx} + Py\right]}_{\frac{d}{dx}[\mu y]} = \mu Q$$

Step 4: Integrate

$$\mu y = \int \mu Q dx + C$$

Step 5: Solve for y:

$$y = \frac{1}{\mu}\int \mu Q dx + C\cdot\frac{1}{\mu}$$

PDE

$$\frac{1}{3}\frac{\partial u}{\partial x} + x^3 y u = 0$$

1. $$\frac{\partial u}{\partial x} + (3x^3 y)u = 0$$

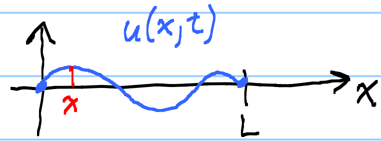
2. $$\mu = e^{\int 3x^3 y\, dx} = e^{\frac{3}{4}x^4 y}$$

3. $$\underbrace{\left(e^{\frac{3}{4}x^4 y}\right)\left(\frac{\partial u}{\partial x} + 3x^3 y u\right)}_{\frac{\partial}{\partial x}\left[e^{\frac{3}{4}x^4 y}\cdot u\right]} = 0$$

4. $$e^{\frac{3}{4}x^4 y} u = C(y)$$

5. $$u = C(y)e^{-\frac{3}{4}x^4 y}$$

Vibrating String Prob:



Wave Eqn: $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Boundary Cond BC: $$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

Initial Cond IC $$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

Separation of variables: $u(x,t) = X(x)T(t)$

PDE $\quad \frac{\partial^2}{\partial t^2}[XT] = c^2 \frac{\partial^2}{\partial x^2}[XT]$

$$XT'' = c^2 X'' T$$

Separate vars: $\quad \underbrace{\frac{X''}{X}}_{\text{fcn of } x} = \underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{fcn of } t} = \lambda$, a const.

X prob: $\quad X'' - \lambda X = 0$

BC $\rightarrow$ $X(0)=0, X(L)=0$

S-L Prob.

Why: $\quad u(L,t) = \underbrace{X(L)}_{\substack{\text{need} \\ =0}} \underbrace{T(t)}_{\text{don't want } T(t)\equiv 0} \overset{\text{want}}{=} 0$

Got e-vals $\quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2$, e-fcns $X_n(x) = \sin\frac{n\pi x}{L}$

T prob: $\quad T'' - c^2\lambda_n T = 0$

$$T'' + \left(\frac{cn\pi}{L}\right)^2 T = 0$$

Get $\quad T_n(t) = A_n \cos\frac{cn\pi}{L}t + B_n \sin\frac{cn\pi}{L}t$

Get $\quad u_n(x,t) = X_n(x)T_n(t)$

Superposition Principle: $\quad u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

Linear PDE

Last thing: IC

ans $\longrightarrow$ $u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(-A_n \frac{cn\pi}{L}\sin\frac{cn\pi t}{L} + B_n\frac{cn\pi}{L}\cos\frac{cn\pi}{L}t\right)$$

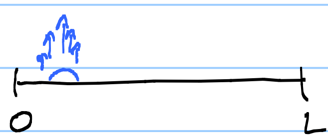
IC: $\quad u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \overset{\text{want}}{=} f(x)$

Aha! F. Sine Series: $\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \underbrace{B_n\frac{cn\pi}{L}}_{\text{F. Sine Series coeff for } g(x)} \sin\frac{n\pi x}{L} \overset{\text{want}}{=} g(x)$$

$$\boxed{B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

Piano prob:



$f(x)$ = little bump

$g(x)$ = little bump

Next time: Use trig ident.

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

Johann's Sol^n $\longrightarrow$ Jacob's Sol^n