Lesson 37 more Vibrating string

PDE $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ IC $\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$ BC $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$

Got $u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$ ← F. Sine Series for f

$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$

Case: $f(x)$ given, but $g(x) \equiv 0$. (Plucked)

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L}\right)$$

$\sin(\alpha)\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct)$$

Aha! $= \boxed{\frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)}$

Really? $u_t = \frac{1}{2}f'(x+ct)(c) + \frac{1}{2}f'(x-ct)(-c)$

$u_{tt} = \frac{1}{2}f''(x+ct)c^2 + \frac{1}{2}f''(x-ct)(-c)^2$

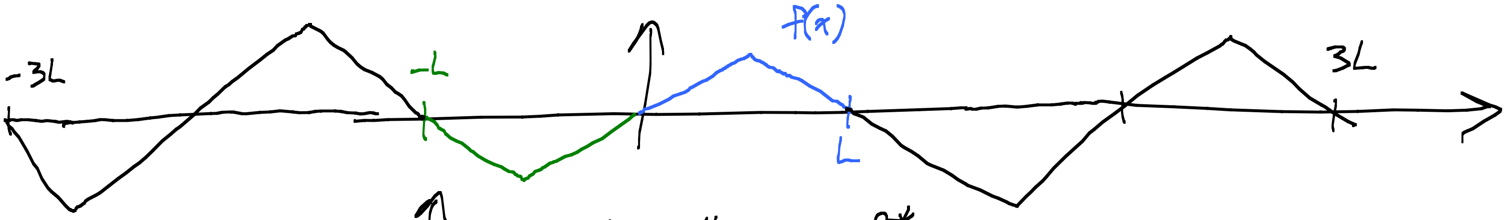
$= c^2 u_{xx}$ ✓

$u(x,0) = \frac{1}{2}f(x+0) + \frac{1}{2}f(x-0) = f(x)$ ✓

$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}f'(x)c - \frac{1}{2}f'(x)c = 0$ ✓

$u(0,t) = \frac{1}{2}f(ct) + \frac{1}{2}f(-ct) \stackrel{?}{=} 0$ Yes!

Important: The f in the formula is the sum of a F. Sine series for given f on $[0,L]$. So it is the odd periodic extension of given f.



The book calls this f^*

General case where $g(x) \not\equiv 0$:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

where $G(x) = \int_0^x g(u)\,du$, and g is the odd periodic extension of given g on $[0,L]$.

D'Alembert's sol^n: $c^2 u_{xx} = u_{tt}$

New variables: $v = x + ct$
$w = x - ct$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}$$

$$= u_v \cdot 1 + u_w \cdot 1 = u_v + u_w$$

$$u_{xx} = (u_v + u_w)_v \cdot \underbrace{\frac{\partial v}{\partial x}}_{1} + (u_v + u_w)_w \cdot \underbrace{\frac{\partial w}{\partial x}}_{1}$$

$$= u_{vv} + 2u_{wv} + u_{ww}$$ ← assuming that u is $\mathcal{C}^2$ smooth so that $u_{wv} = u_{vw}$

Get $u_{tt} = c^2\left(u_{vv} - 2u_{wv} + u_{ww}\right)$

Plug into wave eqn. Get $4c^2 u_{wv} = 0$

$$\boxed{u_{wv} \equiv 0}$$

$$\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial v}\right] \equiv 0$$

$\frac{\partial u}{\partial v}$ is a fcn of v alone!

So $\quad \frac{\partial u}{\partial v} = C(v)$

and so $\quad u = \int_0^v C(\mu)\,d\mu + K(w)$

Aha! $\quad u = \varphi(v) + \psi(w)$

$$\boxed{u = \varphi(x+ct) + \psi(x-ct)}$$

$$\frac{\partial u}{\partial t} = \varphi'(x+ct)c - \psi'(x-ct)c$$

Want $\quad u(x,0) = \varphi(x) + \psi(x) \overset{\text{want}}{=} f(x) \qquad (A)$

$$\frac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) \overset{\text{want}}{=} g(x) \qquad (B)$$

Integrate (B): $\quad c\varphi(x) - c\psi(x) = \int_0^x g(\mu)\,d\mu$

$$\varphi(x) - \psi(x) = \underbrace{\frac{1}{c}\int_0^x g(\mu)\,d\mu}_{G(x)} \qquad (B)'$$

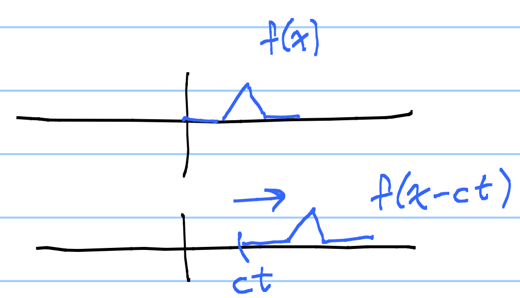
assuming $\varphi(0)=0$, $\psi(0)=0$.

$(A)+(B)' = \quad 2\varphi(x) = f(x) + G(x)$

$(A)-(B)' = \quad 2\psi(x) = f(x) - G(x)$

Get Jacob's solⁿ!

$f(x-\underbrace{ct}_{\text{shift}})$



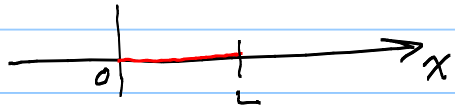
$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

← left $\qquad$ right →

Maple demo here

Monday: Hot wire problem with <u>insulated</u> ends.

Kelvin's law: (Heat flow) $= -c$ (Temp grad)



$u(x,t) = $ temp.

Heat eqn., BC: $\begin{cases} \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 0 \end{cases}$, I.C. $u(x,0) = f(x)$

Try $\quad u(x,t) = X(x)\,T(t)$

B.C. $\quad \frac{\partial u}{\partial x}(x,t) = X'(x)\,T(t)$

Need $\quad \frac{\partial u}{\partial x}(0,t) = \underline{X'(0)}\,T(t) = 0$

$\quad\quad \frac{\partial u}{\partial x}(L,t) = \underline{X'(L)}\,T(t) = 0$

Sturm-Liouville Problem: $\quad X'' - \lambda X = 0$

$\quad\quad X'(0) = 0, \; X'(L) = 0$