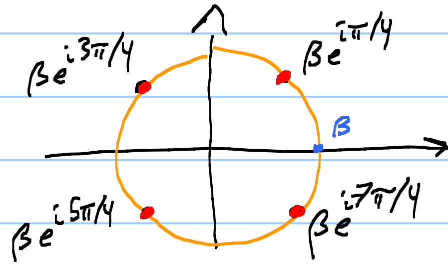
Lesson 39 on Heat problems   WebEx Off. Hr. Tues 8-9pm   (36, 37, 38 due Wed.) 1

15, 16. $X^{(4)} + \lambda X = 0$ Case $\lambda > 0$: $\lambda = \beta^4$: $r^4 + \beta^4 = 0$

Roots $\beta e^{i\pi/4}, \beta e^{i3\pi/4}, \beta e^{i5\pi/4}, \beta e^{i7\pi/4}$

$= \beta\left(\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i\right)$



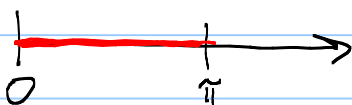
$$X(x) = c_1 e^{\beta x/\sqrt{2}} \cos\frac{\beta x}{\sqrt{2}} + c_2 e^{\beta x/\sqrt{2}} \sin\frac{\beta x}{\sqrt{2}} + c_3 e^{-\beta x/\sqrt{2}} \cos\frac{\beta x}{\sqrt{2}} + c_4 e^{-\beta x/\sqrt{2}} \sin\frac{\beta x}{\sqrt{2}}$$

BC $X(0) = 0$ $X''(0) = 0$

$X(L) = 0$ $X''(L) = 0$

Four equs in 4 c's. Only get zero sol^n.

Heat prob:



$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$u(x,0) = f(x)$ ← given

IC

insulated   iced

$\frac{\partial u}{\partial x}(0,t) = 0$ $u(\pi,t) = 0$

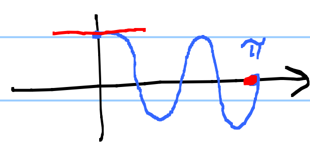
BC

$u(x,t) = X(x)T(t)$

$$\boxed{\begin{gathered} X'' + \lambda X = 0 \\ X'(0) = 0,\ X(\pi) = 0 \end{gathered}}$$

e-vals $\lambda_n = -(n + \tfrac{1}{2})^2$ $n = 0, 1, 2, 3$

$X_n(x) = \cos(n + \tfrac{1}{2})x$



$T_n(t) = e^{-(n+\frac{1}{2})^2 c^2 t}$

Gen^l Sol^n : $u(x,t) = \sum_{n=1}^{\infty} c_n \cos(n + \tfrac{1}{2})x \, e^{-(n+\frac{1}{2})^2 c^2 t}$

Last: IC : $u(x,0) = \sum_{n=0}^{\infty} c_n \cos(n + \tfrac{1}{2})x \overset{\text{want}}{=} f(x)$

S-L Theory: $\cos(n+\frac{1}{2})x$ are $\perp$ on $[0,\pi]$ and they are complete!

How to find c_n's: Mult by $\cos(m+\frac{1}{2})x$ and $\int_0^\pi$:

$$\int_0^\pi f(x)\cos(m+\tfrac{1}{2})x\,dx = c_m\int_0^\pi \cos^2[(m+\tfrac{1}{2})x]\,dx + \Sigma' 0\text{'s}$$

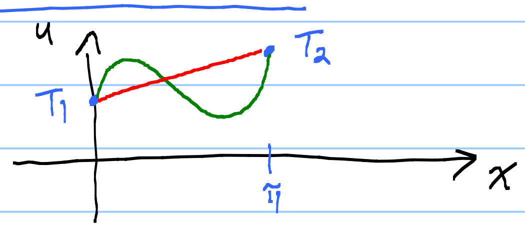
$$= c_m \int_0^\pi \tfrac{1}{2}\left(1+\cos(2m+1)x\right)dx$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= c_m \cdot \frac{\pi}{2}$$

$$\boxed{c_m = \frac{2}{\pi}\int_0^\pi f(x)\cos(m+\tfrac{1}{2})x\,dx}$$

What if we wanted $u(0,t)=T_1$ and $u(\pi,t)=T_2$?



Ouch! We need homog BC so we can add sol^n's to get more sol^n's.

Classic trick: What is the Steady State sol^n?

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$t\to\infty \quad \downarrow$$

$$0 = c^2\frac{d^2}{dx^2}u$$

So $u = c_1 x + c_2$

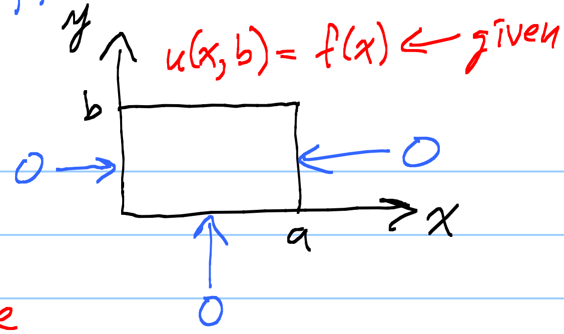
Get $\underbrace{u = T_1 + (T_2 - T_1)\frac{x}{L}}_{u_\infty(x)}$

Big idea: Let $U(x,t) = u(x,t) - u_\infty(x)$

Aha! U satisfies our favorite homog BC! We have a formula for U.

Get a formula $u(x,t)$.

Hot rectangular plate:



$$\frac{\partial u}{\partial t} = c^2 \Delta u$$

$$0 = c^2 \Delta u$$

steady state
$t \to \infty$

Laplace Eqn $\Delta u = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Separation of vars
$u(x,y) = X(x)\,Y(y)$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda, \text{ const.}$$

BC

Bottom: $u(x,0) = X(x)\underbrace{Y(0)}_{0} \overset{\text{want}}{=} 0$

Left: $u(0,y) = \underbrace{X(0)}_{0}\,Y(y) = 0$

Right: $u(a,y) = \underbrace{X(a)}_{0}\,Y(y) = 0$

$$\boxed{\begin{array}{c} X'' - \lambda X = 0 \\ X(0) = 0,\ X(a) = 0 \end{array}} \qquad \boxed{\begin{array}{c} Y'' + \lambda Y = 0 \\ Y(0) = 0 \end{array}}$$

Hard — Easy

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2$$

$$\boxed{X_n(x) = \sin\frac{n\pi x}{a}}$$

$$Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0$$

$$r = \pm\frac{n\pi}{a}$$

$$Y_n(y) = c_1 \cosh\frac{n\pi y}{a} + c_2 \sinh\frac{n\pi y}{a}$$

$$Y_n(0) = c_1 \cdot 1 + c_2 \cdot 0 \overset{\text{want}}{=} 0$$

So $c_1 = 0$. Take $c_2 = 1$

Homog BC: Can add sol^{n}s and get more:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}}$$

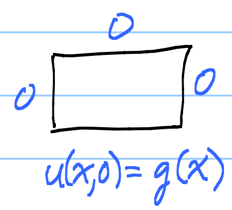
Last thing: Top: $u(x,b) = \sum_{n=1}^{\infty} \Big(\underbrace{c_n \sinh\frac{n\pi b}{a}}\Big) \sin\frac{n\pi x}{a} \stackrel{\text{want}}{=} f(x)$

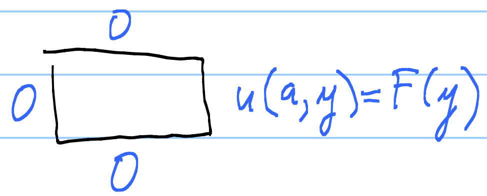
A_n, the F. Sine Series coeff for f!

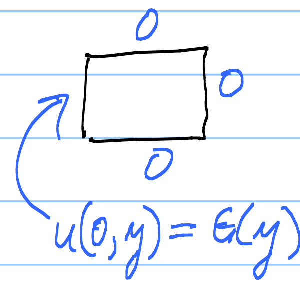
$$\boxed{c_n = \frac{1}{\sinh\frac{n\pi b}{a}} \cdot \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\,dx}$$

Good news: $\frac{1}{\sinh\frac{n\pi b}{a}} \longrightarrow 0$ really fast as $n \to \infty$.

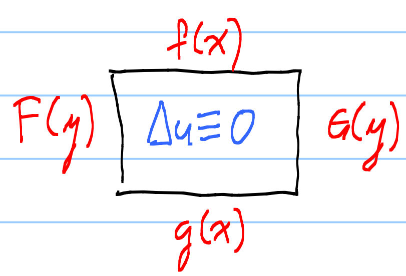
Do 3 more prob:







Famous Problem: Dirichlet Problem



Answer: Sum of 4 sol^{n}s.

Beam problem: Find $X_n(x)$ for certain β's

$$X(x) = c_1 \cos\beta x + c_2 \sin\beta x + c_3 \cosh\beta x + c_4 \sinh\beta x$$

$X(0)=0 \qquad X''(0)=0$

$X(L)=0 \qquad X''(L)=0$

Find e-vals $\lambda = -\beta^4$ that allow non-zero sol'ns.

Get e-fcns $X_n(x)$.

T-prob. $T_n(t) = A_n \cos \mu_n t + B_n \sin \mu_n t$

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) \left[A_n \cos \mu_n t \quad B_n \sin \mu_n t \right]$$

Get A's from initial shape

B's = 0 from rest.