Lesson 4 on 7.5, 7.6

$$A \sim \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} \text{col } 1 & & \text{col } 3 & & \text{col } 5 & \end{array} \end{array}$$

← Row echelon form (non-zero rows always lin. ind.)

$\text{Rank}(A) = 3$

$[1\ 2\ 3\ 4\ 5\ 6]$, $[0\ 0\ 2\ 3\ 4\ 5]$, $[0\ 0\ 0\ 0\ 3\ 4]$

form a basis for the $\underbrace{\text{span of the row vectors of } A}_{\text{Row space of } A}$

$\text{Dim}(\text{Row space } A) = \text{Rank}(A)$.

Span of the column vectors of A = "Column space of A"

Kill 2 prob with E! Basis for col space : $\vec{v}_1, \vec{v}_3, \vec{v}_5$ cor. to bound col's.

$A = [\vec{v}_1, \dots, \vec{v}_6]$.

Or transpose A to make columns rows and $\sim E$.

Facts: $\text{Rank}(A) = \text{Rank}(A^T)$ ← $\text{Dim Row space } A = \text{Dim Col space } A = \text{Rank}(A)$

$(AB)^T = B^T A^T$

EX: $A\vec{x} = \vec{b}$ $\quad [A \mid \vec{b}] \sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 0 & 5 & 6 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right]$ E1, E2

x_1 bound, x_2 bound, x_3 free

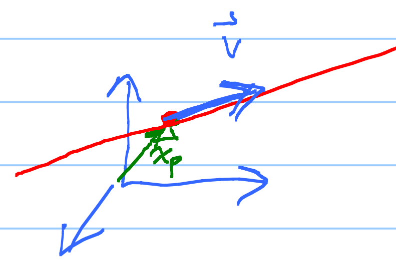
Let $x_3 = t$.

E2: $5x_2 + 6[\underbrace{t}_{x_3}] = 11$: $\boxed{x_2 = \frac{11}{5} - \frac{6}{5}t}$

(Back subst.)

E1: $3x_1 + 2[\underbrace{\tfrac{11}{5} - \tfrac{6}{5}t}_{x_2}] + 1 \cdot [\underbrace{t}_{x_3}] = 10$

$$\boxed{x_1 = \frac{28}{15} + \frac{7}{15}t}$$



$$\begin{cases} x_1 = \frac{28}{15} + \frac{7}{15}t \\ x_2 = \frac{11}{5} - \frac{6}{5}t \\ x_3 = t \end{cases} \qquad \vec{x} = \begin{pmatrix} 28/15 \\ 11/5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7/15 \\ -6/5 \\ 1 \end{pmatrix}$$

$= \vec{x}_p + t\vec{v}$, a line in $\mathbb{R}^3$

Important Fact: Gen^l Sol^n to $A\vec{x} = \vec{b}$ is

$$\vec{x}_p + (t_1\vec{v}_1 + \cdots + t_n\vec{v}_n) \leftarrow S$$

where $\vec{x}_p$ is one particular sol^n to $A\vec{x} = \vec{b}$
and $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$ is the Gen^l Sol^n to homog
syst $A\vec{x} = \vec{0}$.

Why: Step 1: Things in S solve $A\vec{x} = \vec{b}$.

$$\rightarrow A\left[\vec{x}_p + (t_1\vec{v}_1 + \cdots + t_n\vec{v}_n)\right] = \underbrace{A\vec{x}_p}_{\vec{b}} + \sum t_k \underbrace{A\vec{v}_k}_{\vec{0}}$$

$$\vec{b} + \vec{0} = \vec{b} \checkmark$$

Step 2: Must show we get all sol^{n}s this way.

Suppose $\vec{u}$ is a sol^n to $A\vec{x} = \vec{b}$.

Key: $\vec{u} - \vec{x}_p$ solves $A\vec{x} = \vec{0}$

$$A(\vec{u} - \vec{x}_p) = A\vec{u} - A\vec{x}_p = \vec{b} - \vec{b} = \vec{0} \checkmark$$

So $\vec{u} - \vec{x}_p = t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$ for some t's. $\checkmark$

Aha! $\vec{u}$ is in S.

Homogeneous systems $A\vec{x} = \vec{0}$:

Things to note: Rank$(A) \leq$ (# rows of A) = # eqns

Rank(A) + Nullity(A) = # vars = # cols A

Nullity$(A) \leq$ # cols.

EX: $A\vec{x} = \vec{0}$ $\quad A \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Every var bound. Unique sol^n $\vec{x} = \vec{0}$.

Rank$(A) = 3 = $ # vars. Unique sol^n

But replace 5 by 0. Rank$(A) = 2$. x_3 free.
∞ many sol^{n}s.

Fact: # vars < # eqns. Homog prob can
have unique sol^n or ∞ many.

EX: $A \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 2 bound
2 free

∞ many sol^{n}s. [3 eqns, 4 vars]

Fact: Homog sys always has ∞ many sol^{n}s if too many vars.

why: $\text{Rank}(A) \le \#\text{eqns} < \#\text{vars}$

free vars exist!

7.4: 32. $\left\{ \vec{v} \in \mathbb{R}^3 \;\; (*) \begin{cases} 3v_1 - 2v_2 + v_3 = 0 \\ 4v_1 + 5v_2 = 0 \end{cases} \right\} = S$

Subspace test: 1) Suppose (v_1, v_2, v_3) and (w_1, w_2, w_3) solve $(*)$. Show that $(v_1+w_1, \dots, v_3+w_3)$ does too.

2) Also check that (cv_1, cv_2, cv_3) does.

Hmmm. $\left[\begin{array}{ccc|c} 3 & -2 & 1 & 0 \\ 4 & 5 & 0 & 0 \end{array}\right]$, $A\vec{v} = \vec{0}$

7.6 1×1 $\det([a]) = a$

2×2 $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a \det([d]) - b \det([c])$

$= ad - bc$

3×3 $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \det(A_{11}) - b \det(A_{12})$

$+ c \det(A_{13})$

4×4 EXPAND Along row 1.