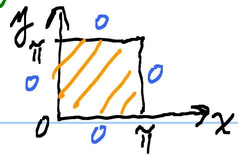
Lesson 41 on Vibrating square drum 12.9

39, 40, 41 due Wed. HWK 12

WebEx Off. Hr. Tues. 8-9 pm

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$



$$u(x,y,t) = X(x)Y(y)T(t)$$

$$\frac{T''}{c^2 T} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}} = \mu$$

T-prob: $T'' - c^2 \mu T = 0$
no BC

$$\frac{X''}{X} = \mu - \frac{Y''}{Y} = \lambda, \text{ a const.}$$

X-prob: $\begin{cases} X'' - \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{cases}$

Classic S-L Prob. e-vals: $\lambda_n = -n^2$

e-fcns: $X_n(x) = \sin nx$

Y-prob: $\mu - \frac{Y''}{Y} = \lambda = -n^2$

$$\begin{cases} Y'' - (\mu + n^2) Y = 0 \\ Y(0) = 0,\ Y(\pi) = 0 \end{cases}$$

Only time get non-zero sol^{n}s: $\boxed{\mu + n^2 = -m^2}$ ← $m = 1, 2, 3, \ldots$

Non-zero sol^n is $Y_m(y) = \sin my$.

$$\boxed{\mu = -(n^2 + m^2)}$$

T-prob: $T'' - c^2\left[-(n^2+m^2)\right] T = 0$

$$T_{nm}(t) = A_{nm} \cos c\sqrt{n^2+m^2}\, t + B_{nm} \sin c\sqrt{n^2+m^2}\, t$$

Get $u_{nm}(x,y,t) = \sin nx \sin my \, T_{nm}(t)$

Gen'l Sol'n: $u(x,y,t) = \sum_{n,m=1}^{\infty} u_{nm}(x,y,t)$

Last step: IC Let's take $u(x,y,0) = f(x,y)$

$$\frac{\partial u}{\partial t}(x,y,0) = 0$$

Note: Get $\frac{\partial u}{\partial t}(x,y,0) \equiv 0$ by taking all B_{nm}'s $= 0$.

So $u(x,y,t) = \sum_{n,m=1}^{\infty} A_{nm} \sin nx \sin my \cos(\sqrt{n^2+m^2}\, t)$

Want $u(x,y,0) = \sum_{n,m=1}^{\infty} A_{nm} \sin nx \sin my \overset{\text{want}}{=} f(x,y)$

Double Fourier Series! $\varphi_{nm} = \sin nx \sin my$ are $\perp$ on $[0,\pi] \times [0,\pi]$:

$$\langle \varphi, \psi \rangle = \iint_{\square} \varphi \psi \, dA = \int_0^{\pi} \int_0^{\pi} \varphi \psi \, dx \, dy$$

$f = \sum A_{nm} \varphi_{nm}$ ← mult by φ_{NM}

$$\iint f \varphi_{NM} \, dA = \sum A_{nm} \iint \varphi_{nm} \varphi_{NM} \, dA$$

$$= A_{NM} \underbrace{\int_0^{\pi} \int_0^{\pi} \varphi_{NM}^2 \, dx \, dy}_{\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}}$$

$$A_{NM} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x,y) \sin nx \sin my \, dx \, dy$$

L42. p. 584: 4, 5, 7 just computing A_{nm}'s.

: 18. Pitch of drum

lowest freq: $c\sqrt{1^2+1^2}$

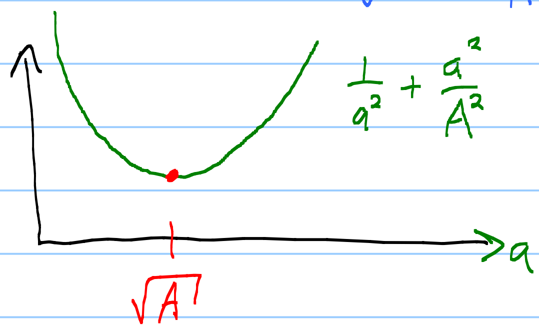
What lowest drum of fixed area $A = ab$

$$u(x,t,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{a} \sin\frac{n\pi y}{b} \cos c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

lowest freq: $c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \omega$

$A = ab \qquad b = \frac{A}{a}$

$$\omega = c\pi\sqrt{\frac{1}{a^2} + \frac{a^2}{A^2}}$$



Lowest rec. drum of area A is square!

Nodal lines. $65 = n^2 + m^2$ in more than two ways!

$$\sum_{n^2+m^2=65} c_{nm} \sin nx \sin my \quad \text{zero set nodal lines.}$$

Euler Equations: $ax^2y'' + bxy' + cy = 0$

Try $y = x^p \qquad y' = px^{p-1} \qquad y'' = p(p-1)x^{p-2}$

Plug in: $ap(p-1)x^p + bpx^p + cx^p = 0$ (want)

$$[ap(p-1) + bp + c]\,x^p = 0$$

need $= 0$

Char. Poly. Get two roots p_1, p_2.

Case 1: p_1, p_2 real $\neq$.

$$y = c_1 x^{p_1} + c_2 x^{p_2}$$

Case 2: $p = a \pm bi$

Take $p = a + bi$

Get complex sol^n $y = x^{a+bi} = x^a x^{bi}$

$= x^a \left(e^{\ln x}\right)^{bi}$

$= x^a e^{b \ln x\, i}$

$= x^a \left(\cos b\ln x + i \sin b \ln x\right)$

$= \left(x^a \cos b \ln x\right) + i \left(x^a \sin b \ln x\right)$

two real sol^{n}s!

Gen^l Sol^n $y = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \ln x)$

Case 3: p_1 repeated root.

Get $y_1 = x^{p_1}$.

Need y_2. Classic method: Try $y_2 = u x^{p_1}$.
Reduction of order

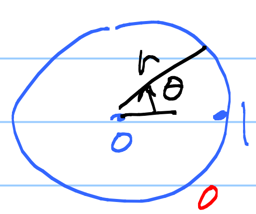
Plug into ODE and try to make it work.

Need to solve a first order ODE in u.

Sol^n is $u = \ln x$. Get $y_2 = x^{p_1} \ln x$

Gen^l Sol^n $y = c_1 x^{p_1} + c_2 x^{p_1} \ln x$

Circular vibrating membrane:



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$$= c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

Sep of var: $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

Invisible BC: $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$ periodic BC

$$\Theta'' - \lambda \Theta = 0$$

S-L Prob. Sol^n is <u>Full</u> Fourier Series.