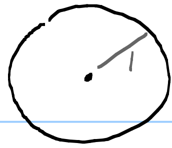
Lesson 42 on 12.10 Vibrating circular membrane

39, 40, 41 due Wed.
(42 do but never due)
WebEx Off. Hr. Tues. 8-9 pm



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$$= c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

Take $c=1$

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

$$R\Theta T'' = R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T$$

BC: $u(1,\theta,t) = 0$

divide by $R\Theta T$:

IC: $\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$

$$\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \mu, \text{ a const.}$$

For now, assume u does not depend on θ. So $\Theta'' \equiv 0$.

R-prob: $\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} = \mu$

$$\begin{cases} r^2 R'' + rR' - \mu r^2 R = 0 \\ \text{BC} \quad R(1) = 0 \end{cases}$$

Bessel's Eqn!

Case $\mu = 0$: $\quad r^2 R'' + rR' = 0$ Euler Eqn.

Try $R(r) = r^p$:

$$r^2 \cdot p(p-1)r^{p-2} + r p r^{p-1} \overset{\text{want}}{=} 0$$

$$\Big[\underbrace{p(p-1) + p}_{\text{need}=0}\Big]\, r^p = 0$$

$$p^2 - p + p = p^2 = 0 \quad \boxed{p = 0, 0}$$

Get $R_1(r) = r^p = r^0 = 1$.

$R_2(r) = r^p \ln r = r^0 \ln r = \ln r$ ← drum ripper!

$c_2 = 0$

Get $R(r) = c_1 \cdot 1$. Need $R(1) = c_1 \overset{\text{want}}{=} 0$

Ouch. $c_1 = 0$ too!

Only get zero soln.

Case $\mu > 0$: Can show there are no non-zero bounded sol^{n}s.

Case $\mu < 0$: Write $\mu = -k^2$

$$r^2 R'' + rR' + k^2 r^2 R = 0, \quad R(1) = 0$$

To put in Standard Form, change vars: $s = kr$.

$$(*) \qquad s^2 \frac{d^2R}{ds^2} + s\frac{dR}{ds} + s^2 R = 0 \leftarrow \text{Bessel's Eqn of order } 0$$

How to find sol^{n}: Try power series:

$$R(s) = a_0 + a_1 s + a_2 s^2 + \cdots$$
$$R'(s) = a_1 + 2a_2 s + 3a_3 s^2 + \cdots$$
$$R''(s) = 2a_2 + 3\cdot 2 a_3 s + \cdots$$

Plug into $(*)$ and collect coeff of s^n.

$$\underbrace{(\quad)}_{\text{must}=0} + \underbrace{(\quad)}_{=0}s + \underbrace{(\quad)}_{=0}s^2 + \cdots \overset{\text{want}}{=} 0$$

Can recursively solve this. [Undet Coeff. Forever.]

Get Bessel Function of order 0 of the first kind.

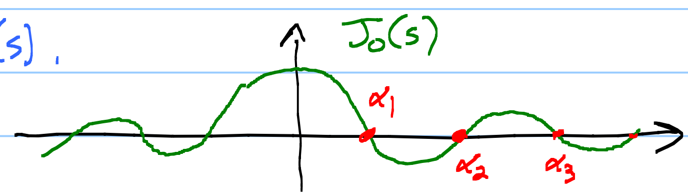
$J_0(s)$ = power series with $(n!)^2$ in denom!

Sol^{n}: $\quad R(s) = c_1 J_0(s) + c_2 \underbrace{Y_0(s)}$

second sol^{n} blows up at $s=0$. so $c_2 = 0$.

Take $c_1 = 1$.

$R(s) = J_0(s)$.



Unchange vars: $\boxed{R(r) = J_0(kr)}$

BC: $R(1) = J_0(k \cdot 1) = 0$

Need $k = \alpha_n$, $n = 1, 2, 3, \ldots$

$\mu = -k^2 = -\alpha_n^2$ ← e-vals

E-fcns: $R_n(r) = J_0(\alpha_n r)$

T-prob: $T'' - \mu T = 0$

$\mu = -\alpha_n^2$

$T'' + \alpha_n^2 T = 0$

$T_n(t) = A_n \cos \alpha_n t + B_n \sin \alpha_n t$

freq of a mode of oscillation

Get $u(r,t) = \sum_{n=1}^{\infty} \left(A_n J_0(\alpha_n r) \cos \alpha_n t + B_n J_0(\alpha_n r) \sin \alpha_n t \right)$

Last thing: IC $u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \overset{\text{want}}{=} f(r)$

$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} \alpha_n B_n J_0(\alpha_n r) \overset{\text{want}}{=} g(r)$

Take $g(r) \equiv 0$. "Pluck" the drum.

Huge fact: S-L Theory. $J_0(\alpha_n r)$ are $\perp$ on $[0,1]$. (weight fcn $= r$)

$$\int_0^1 f(r)\, r J_0(\alpha_m r)\, dr = \sum_{n=1}^{\infty} A_n \int_0^1 J_0(\alpha_n r)\, r J_0(\alpha_m r)\, dr$$

$$= A_m \int_0^1 r J_0(\alpha_m r)^2\, dr$$

Solve for A_m.

Getting Θ back in is fun. $\Theta'' - \lambda \Theta = 0$

periodic BC $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$

Standard Fourier Series