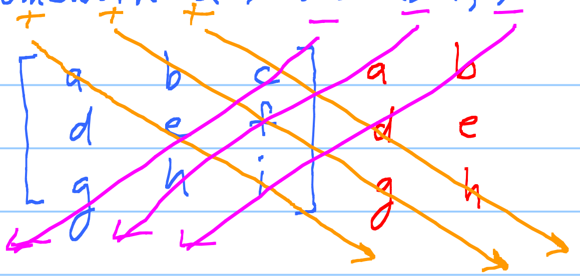
Lesson 5 on 7.7, 7.8

Homework 2: Lessons 4,5,6 due Wed. after Labor Day



MA 511 Linear Alg.

MA 527: $\det(\underset{\uparrow\, n\times n}{A})$ defined inductively:

$\det[a_{11}] = a_{11}$, $\det[n\times n]$ = expand along row 1 to reduce to previous size.

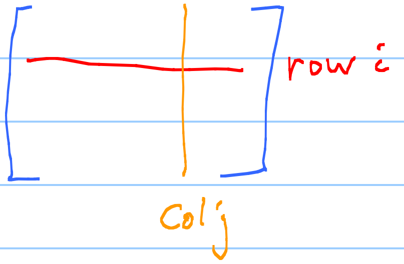
$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-1)\, b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + (+1)\, e \det\begin{bmatrix} a & c \\ g & i \end{bmatrix} + (-1)\, h \det\begin{bmatrix} a & c \\ d & f \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

expanded along col 2.

Words:

$A =$



A_{ij} = matrix from A by crossing out row i and col j

$M_{ij} = \det(A_{ij})$ = "ij minor"

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix} = \left[(-1)^{i+j}\right]$$

$C_{ij} = (-1)^{i+j} M_{ij}$ = "ij cofactor"

Fact: $\det(\underset{\uparrow\, n\times n}{A}) = \sum_{j=1}^{n} a_{ij}\, C_{ij}$ ← i fixed, expand along row i

$= \sum_{i=1}^{n} a_{ij} C_{ij}$ ← j fixed

expand along col j

Properties: 1) $\det(I) = 1$

2) Switching rows changes sign of det

3) Adding c times one row to another does NOT change the det.

4) Multiplying a row by c multiplies det by c. [Can factor out constants from rows.]

5) $\det(A^T) = \det(A)$

6) #5 ⟹ 2, 3, 4 have column versions.

7) $\det(\underbrace{A\,B}_{\text{both } n\times n}) = \det(A)\det(B)$

if and only if

8) $\det(A) \neq 0$ iff $\underbrace{\underbrace{\text{Rank}(A) = n}_{A\vec{x}=\vec{0} \text{ has only } \vec{x}=\vec{0} \text{ sol}^n}}_{\text{rows of } A \text{ are indep}}$

9) $\det(A) = 0$ iff $\underbrace{\underbrace{\text{Rank}(A) < n}_{A\vec{x}=\vec{0} \text{ has } \infty \text{ many sol}^n\text{s}}}_{\text{rows } A \text{ dep.}}$

Adult way to calc. det.

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 3 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & -5 & -8 \end{bmatrix} -3R_1 + R_3$$

Kill Kill

$$= \det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & -7/4 \end{bmatrix} \tfrac{5}{4}R_2 + R_3$$ ← upper Δ!

$$= (1)(4)\left(-\frac{7}{4}\right) = -7$$

Fact: det(upper Δ) = product down diag

Why: $$\det \underset{\uparrow}{\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}} = a \det \underset{\uparrow}{\begin{bmatrix} b & * \\ 0 & c \end{bmatrix}} = a\left(b \det[c]\right) = abc \checkmark$$

Inverse of a matrix:

$$A\underbrace{[\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n]}_{A \cdot A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}}_{I} \qquad \vec{e}_j = j\text{-th col}$$

$$A\vec{x}_j = \vec{e}_j \qquad [A \mid \vec{e}_j] \rightsquigarrow [I \mid \vec{y}_j]$$

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 \\ \vdots \\ x_n = y_n \end{cases}$$

Aha! $\vec{x}_j = \vec{y}_j$

Aha! $[A \mid I] \rightsquigarrow [I \mid A^{-1}]$

Gauss-Jordan: EX:

$$\left[\begin{array}{ccc|ccc} \overbrace{1 \quad 2}^{A} & & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right]$$

Rank must be 3

det ≠ 0

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -4/3 & 1 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad}_{A^{-1}}$$

Using A^{-1} to solve $A\vec{x} = \vec{b}$ ← multiply on left by A^{-1}

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$\underbrace{(A^{-1}A)}\vec{x} = A^{-1}\vec{b}$$

$$\begin{array}{c} I \\ I\vec{x} \\ \shortparallel \\ \vec{x} \end{array} = A^{-1}\vec{b}$$

Famous formula for A^{-1}:

$$A^{-1} = \frac{1}{\det(A)} \underbrace{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{T}}_{\text{classical adjoint}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Cramer's Rule: $\underset{\uparrow n\times n}{A}\vec{x} = \vec{b}$ $\quad \text{Rank}(A) = n \quad (\det(A) \neq 0)$

Let A_j = matrix from A by replacing col j by $\vec{b}$

Sol^{n} $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ where $x_j = \dfrac{\det(A_j)}{\det(A)}$

Words: $\det(\underset{\uparrow n\times n}{A}) = 0$ Bad! A is singular

$\text{Rank}(A) < n$, rows A dependent.

2) $\det(A) \neq 0$ Good! A is non-singular

A^{-1} exists!