Lesson 6 on 7.9 Vector Spaces (No lecture on Monday. Lessons 4,5,6 due Wed.)

EX: $y'' - y = 0$ $\quad L[y] = y'' - y$. Try $L[e^{rx}] = (r^2-1)e^{x} = 0$ want

linear trans.

$r = \pm 1$

Gen^l Sol^n $\quad y = c_1 e^{x} + c_2 e^{-x} \leftarrow \text{span}(e^{x}, e^{-x})$

$C^2(\mathbb{R})$ = twice cont. diff'ble fcns on $\mathbb{R}$.

Vector space: Box on p. 310.

Q: Are e^{x}, e^{-x} lin. indep.?

Suppose $c_1 e^{x} + c_2 e^{-x} \equiv 0 \leftarrow$ zero vector

Case $c_1 \neq 0$: $\quad c_1 e^{x} = -c_2 e^{-x}$

No way! $\rightarrow e^{2x} = \dfrac{-c_2}{c_1} \leftarrow c_1 \neq 0$

Case $c_1 = 0$: $\quad 0 \cdot e^{x} + c_2 e^{-x} \equiv 0$

$c_2 e^{-x} = 0$

e^{a} is never 0

So c_2 must be zero, too.

See $c_1 = 0$, $c_2 = 0$. They are indep. They form a basis.

Interesting fact: $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$, $\sinh x = \dfrac{e^{x}-e^{-x}}{2}$

form a basis too!

EX: $S' = \{ f \in C^2(\mathbb{R}) \text{ with } f(x) \geq 0, \forall x \}$

Q: Is S' a vector space?

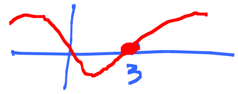
Subspace Test: Only need to check

1) closed under + ✓

2) closed under mult. by c $\quad$ No!

EX: $f(x) = x^2$. $c = -1$. Oops! $-x^2$ is out.

EX: $\{ f \in C^2(\mathbb{R}) : f(3) = 0 \}$



Is a subspace of $C^2(\mathbb{R})$.

EX: M_{nm} = all $n \times m$ matrices

$$M_{22} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{basis}} \right\}$$

EX: P_n = polys of deg $\leq n$

$P_3 = \text{span}(1, x, x^2, x^3) \leftarrow$ basis!

Indep? $\quad \underbrace{c_0 + c_1 x + c_2 x^2 + c_3 x^3}_{Q(x)} \equiv 0$

$$\frac{d^3 Q}{dx^3} = 3!\, c_3 \checkmark, \quad \frac{d^2 Q}{dx^2} = 2!\, c_2 \checkmark \quad \ldots$$

Inner product space: $\quad \vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n$

$$= \vec{a}^T \vec{b}$$

$$= [a_1, \ldots, a_n] \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Properties: Box on p. 312

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$(\vec{a}_1 + \vec{a}_2) \cdot \vec{b} = \vec{a}_1 \cdot \vec{b} + \vec{a}_2 \cdot \vec{b}_2$, $\quad (c\vec{a}) \cdot \vec{b} = c\, \vec{a} \cdot \vec{b}$

$$\vec{a} \cdot \vec{a} \begin{cases} > 0 & \text{if } \vec{a} \neq \vec{0} \\ = 0 & \text{if } \vec{a} = \vec{0} \end{cases}$$

Norm: $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + \cdots + a_n^2}$

$\vec{a} \cdot \vec{b} = 0 \leftarrow \vec{a} \perp \vec{b}$

Great thing! Orthog basis $\vec{u}_1, \ldots, \vec{u}_n$

$\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$. $\quad (\vec{u}_k \neq \vec{0}, \forall k)$

Suppose $\left(\vec{b} = c_1 \vec{u}_1 + \cdots + c_n \vec{u}_n \right) \cdot \vec{u}_k$

$$\vec{u}_k \cdot \vec{b} = \underbrace{c_1 \vec{u}_1 \cdot \vec{u}_k + \cdots + c_n \vec{u}_n \cdot \vec{u}_k}_{\text{all } = 0 \text{ except } c_k \vec{u}_k \cdot \vec{u}_k}$$

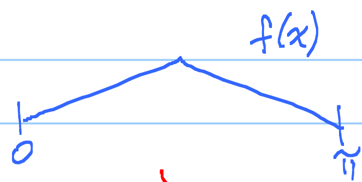
So $\quad c_k = \dfrac{\vec{u}_k \cdot \vec{b}}{\vec{u}_k \cdot \vec{u}_k}$

EX: $C([0,\pi]) \leftarrow$ cont. fcns on $[0,\pi]$

$f \cdot g = ?$ $\qquad f \cdot g = (f,g) = \int_0^{\pi} f(x) g(x)\,dx$

fcns

Johann Bernoulli



$$\sin mx \cdot \left(f(x) = \sum_{n=1}^{\infty} c_n \sin nx \right) \quad \text{(Fourier Series)}$$

$$\int_0^{\pi} \sin nx \, \sin mx \, dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m. \end{cases}$$

$$\int_0^{\pi} f(x) \sin mx \, dx = c_m \underbrace{\int_0^{\pi} \sin^2 mx \, dx}_{\frac{\pi}{2}}$$

Aha! $\quad \boxed{c_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx}$

Linear Transformations: $T: \mathbb{R}^n \to \mathbb{R}^m$

Lect 1: $\quad T\vec{x} = A\vec{x}$

$$A = \underbrace{[T\vec{e}_1, T\vec{e}_2, \ldots, T\vec{e}_n]}_{m \times n}$$

$T \sim A$

$$(T_1 \circ T_2)\vec{x} = T_1 \underbrace{(T_2 \vec{x})}_{A_1 (A_2 \vec{x})}$$

$$= (A_1 A_2)\vec{x}$$

$\sim T_1 \circ T_2$

EX: $T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad T\vec{x} = A\vec{x}$

Inverse? Need

$$\left.\begin{aligned} &\text{Rank}(A) = 3 \\ &\det(A) \neq 0 \\ &A \text{ non-singular} \\ &\text{Dim Row sp} = 3 \\ &\text{Dim Col sp} = 3 \\ &A^{-1} \text{ exists} \end{aligned}\right\} \text{equiv}$$

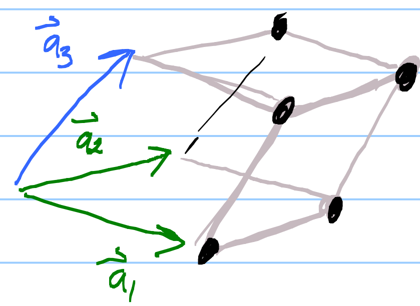
What is T^{-1}? $\quad A\vec{x} = \vec{y}$ ← mult on left by A^{-1}

$$\vec{x} = A^{-1}\vec{y}$$

Aha! $T^{-1} \sim A^{-1}$

<u>Meaning of $\det(A)$</u>: $\quad A = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$

3x3



$$\det(A) = \pm \text{volume}(\text{box})$$

$\det(A) = 0$ ← squashed box (could be a line)