Lesson 7 on 8.1 Eigenvalues & eigenvectors

Note: 2 lowest hwk scores dropped

4, 5, 6 due tonight 11:59 pm on Blackboard.

System: $\begin{cases} \dfrac{dx_1}{dt} = -3x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = x_1 - 2x_2 \end{cases}$

Try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{pmatrix} = \vec{a} e^{\lambda t}$

$\begin{cases} \lambda a_1 e^{\lambda t} = -3a_1 e^{\lambda t} + 2a_2 e^{\lambda t} \\ \lambda a_2 e^{\lambda t} = a_1 e^{\lambda t} - 2a_2 e^{\lambda t} \end{cases}$

$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\boxed{\lambda \vec{a} = \mathbb{A} \vec{a}}$ ← want $\vec{a} \neq \vec{0}$

λ eigenvalue, $\vec{a}$ eigenvector

$\begin{cases} 0 = (-3-\lambda)a_1 + 2a_2 \\ 0 = a_1 + (-2-\lambda)a_2 \end{cases}$

$\begin{bmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

same ← $(\underbrace{\mathbb{A} - \lambda \mathbb{I}}_{\text{must be bad!}}) \vec{a} = \vec{0}$

want $\vec{a} \neq 0$

Need $\boxed{\det(\mathbb{A} - \lambda \mathbb{I}) = 0}$ ← Characteristic eqn.

EX: $\det \begin{bmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{bmatrix} = 0$

$(-3-\lambda)(-2-\lambda) - 2 = 0$

$\lambda^2 + 5\lambda + 4 = 0$

$(\lambda+1)(\lambda+4) = 0$

$\boxed{\lambda = -1, -4}$ e.vals.

For $\lambda = -1$: $(\mathbb{A} - \lambda \mathbb{I})\vec{a} = \vec{0}$

$\lambda = -1$

$\left[\begin{array}{cc|c} -3-(-1) & 2 & 0 \\ 1 & -2-(-1) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ E1

← expect!

a_1 bnd

a_2 free

Let $a_2 = t$. E1 $a_1 - (t) = 0$

$a_1 = t$

List: $\begin{cases} a_1 = t \\ a_2 = t \end{cases}$ $\vec{a} = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ← don't want $t=0$ soln.

Def^n: The eigenspace associated to e-val λ is

$E_\lambda = \text{Null}(\mathbb{A} - \lambda \mathbb{I})$ ← vector space!

= set of all e-vects for λ plus the zero vect.

Above: $E_{\lambda=-1} = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. ← 1 dim^e

For $\lambda = -4$: $\left[\begin{array}{cc|c} -3-(-4) & 2 & 0 \\ 1 & -2-(-4) & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ E1

a_1 bnd $\quad$ a_2 free $\quad$ $a_2 = t$

E1 $a_1 + 2(t) = 0$

$a_1 = -2t$

$\begin{cases} a_1 = -2t \\ a_2 = t \end{cases}$ $\vec{a} = \begin{pmatrix} -2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$E_{\lambda=-4} = \text{span}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-4t}$

EX: $\mathbb{A} = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$

Step 1: $\det(\mathbb{A} - \lambda \mathbb{I}) = 0$

$\det \begin{bmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix} = 0$

$(-3-\lambda)(-1-\lambda) + 2 = 0$

$\lambda^2 + 4\lambda + 5 = 0$

Ouch! $\lambda = -2 \pm i$ ← quad. form.

Step 2: Find complex e.vect.

$$(\mathbb{A}-\underset{\uparrow_{\lambda=-2+i}}{\lambda}\mathbb{I})\vec{a}=\vec{0}$$

For $\lambda=-2+i$:

$$\left[\begin{array}{cc|c} -3-(-2+i) & -1 & 0 \\ 2 & -1-(-2+i) & 0 \end{array}\right]$$

$R_2 = (-1+i)R_1$

$$\left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] E1$$

a_1 bnd a_2 free

Let $a_2 = t$

$E1$: $(-1-i)a_1 - (t) = 0$

$$\begin{cases} a_1 = \frac{1}{-1-i}t = \left(-\frac{1}{2}+\frac{1}{2}i\right)t \\ a_2 = t \end{cases}$$

$\vec{a} = t\begin{pmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 1 \end{pmatrix}$ ← My favorite. $t=2$ $\begin{pmatrix} -1+i \\ 2 \end{pmatrix}$

For $\lambda=-2-i$: ← conjugate of first! $\underbrace{\mathbb{A}\vec{a}=\lambda\vec{a}}_{\text{Take conj.}}$

$\vec{a}$ for conj. is the conj. of e.vect.

Get $\vec{a} = t\begin{pmatrix} -\frac{1}{2}-\frac{1}{2}i \\ 1 \end{pmatrix}$. Fav. $\begin{pmatrix} -1-i \\ 2 \end{pmatrix}$

EX: $\mathbb{A} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$ $\det\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = 0$

$$\lambda^2+4\lambda+4=0$$
$$(\lambda+2)^2=0$$

$\lambda=-2$ is a double root.

Word: $\lambda=-2$ is an e-value of "algebraic multiplicity 2"

For $\lambda=-2$: $(\mathbb{A}-\underset{\uparrow_{\lambda=-2}}{\lambda}\mathbb{I})\vec{a}=\vec{0}$

$$\left[\begin{array}{cc|c} -3-(-2) & 1 & 0 \\ -1 & -1-(-2) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} E1 \\ \\ \end{array}$$

a_2 free

Let $a_2 = t$

$E1: \; a_1 - (t) = 0$

$$\begin{cases} a_1 = t \\ a_2 = t \end{cases} \qquad \vec{a} = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_{\lambda=-2} = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \leftarrow 1 \text{ dim}^{\ell}$$

Word: Dimension of e.space for e-val λ is the "geometric multiplicity of λ"

Note: geom. mult. $\leq$ alg. mult. $\leq n \leftarrow$ $\mathbb{A}$ $n \times n$

$<$ bad. $\mathbb{A}$ is "defective"

$=$ good

Word: Defect = (alg. mult) − (geom. mult)

Anything can happen:

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbb{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbb{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det([\;\;] - \lambda \mathbb{I}) = (1-\lambda)^3$$

$\lambda = 1$ is an e-val of alg. mult 3.

Case $\mathbb{A}$: $(\mathbb{A} - \lambda \mathbb{I})\vec{a} = 0$

$\uparrow_{\lambda=1}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

all a's free!

$$\begin{cases} a_1 = t_1 \\ a_2 = t_2 \\ a_3 = t_3 \end{cases} \qquad \vec{a} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$E_{\lambda=1}$ is 3-dim^{l}. $\text{Span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

Case B: $(A - \lambda I)\vec{a} = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E1: \ a_2 = 0$$

a_2 bnd

a_1, a_3 free

$$\begin{cases} a_1 = t_1 \\ a_3 = t_2 \end{cases}$$

$$\begin{cases} a_1 = t_1 \\ a_2 = 0 \\ a_3 = t_2 \end{cases} \qquad \vec{a} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \leftarrow E_{\lambda=1} \text{ is } 2 \text{ dim}^{l}$$

geom mult $= 2$ $\qquad \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = E_{\lambda=1}$

Defect $= 3 - 2 = 1$. Defective.

Case C: $(A - \lambda I)\vec{a} = \vec{0}$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 1-1 & 1 & 0 & 0 \\ 0 & 1-1 & 1 & 0 \\ 0 & 0 & 1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} E1 & a_2 = 0 \\ E2 & a_3 = 0 \end{matrix}$$

a_1 free $\qquad$ Let $a_1 = t$

$$\begin{cases} a_1 = t \\ a_2 = 0 \\ a_3 = 0 \end{cases} \qquad \vec{a} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow 1 \text{ dim}^{l}$$

geom. mult $= 1$. $\quad$ Defect $= 3 - 1 = 2$

Why defective matrices are bad:

Solving a 3x3 system of ODEs. Want 3 lin. independent sol'ns.

Need to find more sol'ns somehow when matrix is defective.