Lesson 8 on 8.3, 8.5. HWK 3: Lessons 7, 8 due Wed. 11:59pm

Remarks: 1) $(\underset{\substack{\uparrow\\ 2\times 2}}{A} - \underset{\nwarrow \text{e-val}}{\lambda} I)\vec{a} = \vec{0} \rightsquigarrow \left[\begin{array}{cc|c} a & b & 0 \\ 0 & 0 & 0 \end{array}\right]$

Read off e-vect. $\begin{pmatrix} b \\ -a \end{pmatrix}$ or $\begin{pmatrix} -b \\ a \end{pmatrix}$

2) Say $\begin{pmatrix} 7/13 \\ -2/13 \end{pmatrix}$ is e-vect. $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$ nicer one.

Def^n: A $n\times n$, $\mathbb{R}$ | $\mathbb{C}$

Symmetric: $A^T = A$

$$\begin{bmatrix} 1 & \pi & e \\ \pi & -2 & -\sqrt{2} \\ e & -\sqrt{2} & 3 \end{bmatrix}$$

Skew-symmetric: $A^T = -A$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Orthogonal: $A^T = A^{-1}$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \quad \text{rows } \perp$$

$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i$$

Hermetian: $\overline{A}^T = A$

$$\begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$$

↖ diag real

Skew hermetian $\overline{A}^T = -A$

$$\begin{bmatrix} 2i & -1+i \\ 1+i & 3i \end{bmatrix}$$

↖ diag pure imag.

Unitary: $\overline{A}^T = A^{-1}$

rows $\perp$ w.r.t

$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i \overline{v_i}$$

↖ inner prod on $\mathbb{C}^n$

Facts: 1) A symmetric $\Rightarrow$ A has real e-vals.

and e-vects for different e-vals are $\perp$.

2) A skew-symmetric $\Rightarrow$ e-vals pure imaginary.

3) A orthogonal $\Leftrightarrow$ rows of A orthonormal

$\Leftrightarrow$ cols of A orthonormal

$$\vec{u}_i\cdot\vec{u}_j = \begin{cases} 0 & i\neq j \\ 1 & i=j \end{cases} \leftarrow \|\vec{u}_i\| = 1$$

4) A orthog. $\Rightarrow \det(A) = \pm 1$

Why 3: orthog $A^T = A^{-1}$

$$\begin{array}{c} \\ \text{row } i \end{array}\begin{bmatrix} \rule{3em}{0.4pt} \end{bmatrix} \begin{array}{c} \text{col } j \\ \begin{bmatrix} \mid \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\mathbb{A}$ $\qquad \mathbb{A}^{-1} = \mathbb{A}^T \qquad \mathbb{I}$

$$\text{col } j = \text{row } j$$

$$(\text{row } i)\cdot(\text{row } j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Why 4: $\begin{cases} \det(\mathbb{A}\mathbb{B}) = \det(\mathbb{A})\det(\mathbb{B}) \\ \det(\mathbb{A}^T) = \det(\mathbb{A}) \end{cases}$

$\mathbb{A}$ orthog: $\mathbb{A}^T = \mathbb{A}^{-1}$ $\qquad \mathbb{A}\underbrace{\mathbb{A}^{-1}}_{\mathbb{A}^T} = \mathbb{I}$

$$\mathbb{A}\mathbb{A}^T = \mathbb{I}$$

$$\det(\mathbb{A})\underbrace{\det(\mathbb{A}^T)}_{=\det(\mathbb{A})} = \det(\mathbb{I})$$

$$\det(\mathbb{A})^2 = 1$$

Notation: $(\vec{u},\vec{v}) = \vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = [u_1 \cdots u_n]\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

Fact: $\boxed{(\mathbb{A}\vec{u},\vec{v}) = (\vec{u},\mathbb{A}^T\vec{v})}$

Why:
$$\begin{aligned}(\mathbb{A}\vec{u},\vec{v}) &= (\mathbb{A}\vec{u})^T\vec{v} \\ &= (\vec{u}^T\mathbb{A}^T)\vec{v} \\ &= \vec{u}^T(\mathbb{A}^T\vec{v}) = (\vec{u},\mathbb{A}^T\vec{v})\end{aligned}$$

Why are e-vects for $\lambda_1 \neq \lambda_2$ of symm $\mathbb{A}$ $\perp$?

$$\begin{cases} \lambda_1\vec{a}_1 = \mathbb{A}\vec{a}_1 \\ \lambda_2\vec{a}_2 = \mathbb{A}\vec{a}_2 \end{cases}$$

$$\begin{aligned}\lambda_1(\vec{a}_1,\vec{a}_2) &= (\lambda_1\vec{a}_1,\vec{a}_2) = (\mathbb{A}\vec{a}_1,\vec{a}_2) \\ &= (\vec{a}_1,\underset{\mathbb{A}}{\underset{\shortparallel}{\mathbb{A}^T}}\vec{a}_2) = (\vec{a}_1,\mathbb{A}\vec{a}_2) \\ &= (\vec{a}_1,\lambda_2\vec{a}_2) \\ &= \lambda_2(\vec{a}_1,\vec{a}_2)\end{aligned}$$

$$\underbrace{(\lambda_1-\lambda_2)}_{\neq 0}(\vec{a}_1,\vec{a}_2) = 0$$

must be 0

Complex case: $\mathbb{A}$ hermitian

Fact: $\boxed{(\mathbb{A}\vec{u},\vec{v}) = (\vec{u},\bar{\mathbb{A}}^T\vec{v})}$ ← complex inner prod.

Suppose $\lambda\vec{a} = \mathbb{A}\vec{a}$.

$$\lambda \underbrace{(\vec{a}, \vec{a})}_{\substack{\neq 0 \\ > 0}} = (A\vec{a}, \vec{a}) \cdots = (\vec{a}, A\vec{a}) = (\vec{a}, \lambda\vec{a}) = \bar{\lambda}(\vec{a}, \vec{a})$$

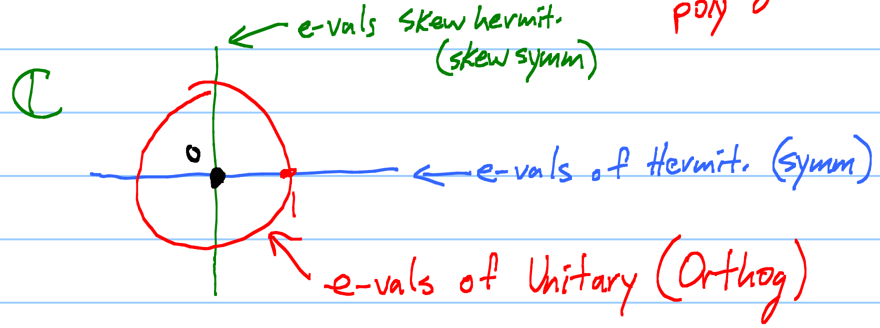
Conclude $\lambda = \bar{\lambda}$. Aha! λ is real.

Note: Symmetric matrices are hermitian.

Fact: Symmetric matrices have real e-vals.

Word: The spectrum of $A_{n \times n}$ is the set of all e-vals of A. [Sol'ns to $\underbrace{\det(A - \lambda I)}_{\text{nth deg poly}} = 0$.]

(MAPLE DEMO HERE)



Fact: Orthogonal matrices preserve length and inner product.

A orthog $\quad \|A\vec{u}\| = \|\vec{u}\|$

$(A\vec{u}, A\vec{v}) = (\vec{u}, \vec{v})$

p. 329: 24. $\quad A^{-1}$ exists $\iff$ e-vals λ_j are all $\neq 0$.

Facts: 1) A^{-1} exists $\iff \det(A) \neq 0$.

2) λ e-val $\iff \det(A - \lambda I) = 0$

Hmmm. $\lambda \neq 0$?

$A\vec{a} = \lambda\vec{a} \leftarrow$ hit with A^{-1} on left.