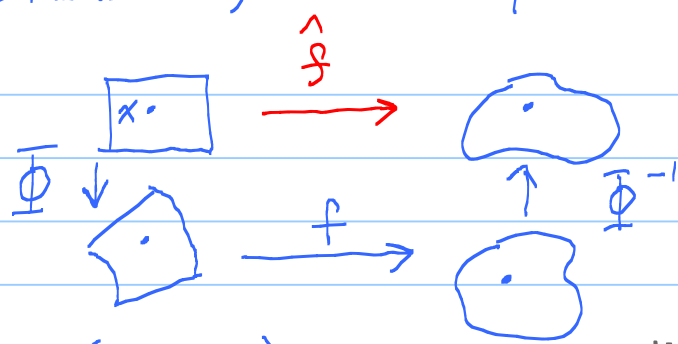
Lesson 9 on 8.4 Diagonalization (Lessons 7, 8 due Wed. HWK 3)

WebEx Office Hours this week T, W 7:30–8:30pm EDT

Change of vars:



Think:

$r = \sqrt{x^2 + y^2}$ $\theta = \arctan \frac{y}{x}$

$x = r\cos\theta$ $y = r\sin\theta$

$$\hat{f}(x) = \Phi^{-1}\Big(f\big(\Phi(x)\big)\Big)$$

$$\hat{f} = \Phi^{-1} \circ f \circ \Phi$$

Linear:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\hat{A}\vec{x}"] \arrow[d, "P\vec{x}"'] & \mathbb{R}^n \\ \mathbb{R}^n \arrow[r, "A\vec{y}"] & \mathbb{R}^n \arrow[u, "P^{-1}\vec{y}"'] \end{tikzcd}$$

$$\boxed{\hat{A} = P^{-1} A P}$$

Similarity transformation

$\hat{A}$ and A are similar.

Prob: Given A, want $\hat{A}$ to be easier.

Wish: $\hat{A}$ = diag. matrix = $\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = D$

$$D\vec{x} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$$

Big fact: If A is real symmetric $n\times n$, then A has a full set of e-vects, meaning there are n lin. indep. e-vects for A. In fact, the basis can be taken to be orthonormal.

[The geom. mult. = alg. mult. for each λ.]

How to do it: A 5×5 symm. $\lambda = \underbrace{1,1,1}_{\text{1 triple root}},2,3$

For $\lambda=1$: $(A-1\cdot I)\vec{a}=\vec{0}$. Get $\vec{a}_1, \vec{a}_2, \vec{a}_3$. indep.

Need to use Gram–Schmidt:

Let $\boxed{\vec{u}_1 = \frac{1}{\|\vec{a}_1\|}\vec{a}_1}$,

$\vec{v}_2 = \vec{a}_2 - (\vec{a}_2,\vec{u}_1)\vec{u}_1$ $\qquad \boxed{\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2}$

$\vec{v}_3 = \vec{a}_3 - (\vec{a}_3,\vec{u}_2)\vec{u}_2 - (\vec{a}_3,\vec{u}_1)\vec{u}_1$

$\boxed{\vec{u}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3}$ $\qquad \vec{u}_1, \vec{u}_2, \vec{u}_3$ orthonormal

same span as $\vec{a}_1, \vec{a}_2, \vec{a}_3$

For $\lambda=2$: Get $\vec{a}_4$. Let $\boxed{\vec{u}_4 = \frac{1}{\|\vec{a}_4\|}\vec{a}_4}$.

Fact: A sym.. So $\vec{u}_4 \perp \vec{u}_1, \vec{u}_2, \vec{u}_3$. !

For $\lambda=3$: Get $\vec{a}_5$. $\boxed{\vec{u}_5 = \frac{1}{\|\vec{a}_5\|}\vec{a}_5}$

Def^n: $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ is called unitary system.

Big idea:
$$A\underbrace{[\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n]}_{P} = [A\vec{u}_1, \dots, A\vec{u}_n]$$
$$= [\lambda_1\vec{u}_1, \dots, \lambda_n\vec{u}_n]$$
$$= \underbrace{[\vec{u}_1, \dots, \vec{u}_n]}_{P}\underbrace{\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n\end{bmatrix}}_{D}$$

Mult by P^{-1} on left:

$\boxed{P^{-1}AP = D}$ $\leftarrow$ Note $P^{-1} = P^T$ when $\vec{u}_1, \dots, \vec{u}_n$ a unitary syst.

Fact: A is diagonalizable $\iff$ A has a full set of e-vects. $P \leftarrow$ cols are the e-vects.

EX:
$$A = \begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{bmatrix}$$

$\det(A-\lambda I) =$

$(+1)(1-\lambda)[(-\lambda)^2-1]$

$= -(\lambda-1)^2(\lambda+1) = 0$

$\lambda = 1, 1, -1.$

For $\lambda = 1$: $(\mathbb{A} - 1\cdot\mathbb{I})\vec{a} = 0$ $\quad \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{E1} \qquad \text{Let } a_2 = t_1,\ a_3 = t_2$$

a_2 a_3 free

E1: $a_1 - (t_2) = 0$. So $\begin{cases} a_1 = t_2 \\ a_2 = t_1 \\ a_3 = t_2 \end{cases}$

$$\vec{a} = t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

orthogonal lucky!

Take $\vec{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ $\quad \|\vec{u}_2\| = 1$

For $\lambda = -1$: $(\mathbb{A} - (-1)\mathbb{I})\vec{a} = 0$ $\quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a_1 = -t \\ \leftarrow a_2 = 0 \end{array}$$

a_3 free $\quad a_3 = t \qquad \begin{cases} a_1 = -t \\ a_2 = 0 \\ a_3 = t \end{cases}$

$\vec{a} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Take $\vec{u}_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ $\perp$ others! $\mathbb{A}$ symm

$$\mathbb{P} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbb{P}^{-1} = \mathbb{P}^T$$

$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathbb{D}$$

λ_1 for $\vec{u}_1$ $\qquad \lambda_3$ for $\vec{u}_3$

Powers become easy: $A = PDP^{-1}$

$$A^m = (PDP^{-1})^m = (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$$

$$= PD^mP^{-1}$$

$$D^m = \begin{bmatrix} \lambda_1^m & & 0 \\ 0 & \lambda_2^m & \\ & & \ddots \quad \lambda_n^m \end{bmatrix}$$

Another great case: A $n \times n$ with n distinct e-vals. Then e-vects form a basis for $\mathbb{R}^n$.
So A is diag'izable. [P^{-1} not as easy to compute because P not unitary syst.]
Fact: e-vects for different e-vals are indep.

Quadratic forms: $Q = 3x_1^2 + 5x_1x_2 + 7x_2^2$

$$Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 5/2 \\ 5/2 & 7 \end{bmatrix}}_{\text{symm. matrix } A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$Q = \vec{x}^T A \vec{x}$, A symm.

Big idea: Change vars $\vec{x} = P\vec{y}$

$$\vec{x}^T = \vec{y}^T \underbrace{P^T}_{P^{-1}}$$

$$\vec{x}^T = \vec{y}^T P^{-1}$$

$$Q = (\vec{y}^T P^{-1}) A P \vec{y}$$

$$= \vec{y}^T \big(\underbrace{P^{-1}AP}_{D}\big)\vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2$$

Is A positive definite? Is $Q(\vec{x}) > 0$ if $\vec{x} \neq 0$?
Aha! Yes, exactly when all λ's are > 0.