

Key

1. Define

$$A = \begin{bmatrix} 1 & 2 & 0 & 8 & 9 \\ 1 & 2 & 0 & 5 & 6 \\ 1 & 2 & 0 & 2 & 3 \end{bmatrix}$$

(5) (a) Use row operations to find an echelon form of A.

$$\begin{pmatrix} 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & -6 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5 pts

Reduced echelon
OK too

$$\begin{pmatrix} 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(5) (b) What is the dimension of the column space of A?

$$\text{dim} = 2$$

5 pts

(10) (c) Find a basis for the space of solutions (the null space) of the system

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

x_2, x_4 free

$$x_4 = 1 \quad x_2 = 0$$

$$\Rightarrow x_3 = -4 \quad x_1 = 8 + 3 = 11$$

$$x_4 = 0 \quad x_2 = 1$$

$$\Rightarrow x_3 = 0 \quad x_1 = -1$$

basis vectors

$$\begin{pmatrix} 11 \\ 0 \\ -4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

5 pts each

2. (20) The matrix $A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has 2 eigenvalues λ_1 and λ_2 . Find λ_1 and λ_2 and basis vectors for the corresponding eigenspaces.

$$\lambda_1 = 1$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_4 free

$$x_2 = 0 \quad x_4 = 1$$

$$\Rightarrow x_3 = 0 \quad x_1 = -4$$

$$x_2 = 1 \quad x_4 = 0$$

$$\Rightarrow x_3 = 0, \quad x_1 = -2$$

$$\lambda_2 = 2$$

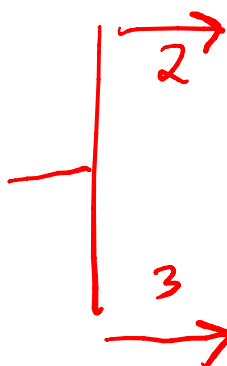
$$\begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

x_1 free

$$\Rightarrow x_4 = 0, \quad x_3 = 0, \quad x_2 = 0$$

5 pts
each

e-vals
5 pts



$$\lambda_1 = 1$$

basis for eigenspace = $\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\lambda_2 = 2$$

basis for eigenspace = $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

3. (20) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$P(\lambda) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 0) + 1(0 + \lambda) \\ = -\lambda^3 + \lambda = \lambda(1 - \lambda^2)$$

$\lambda = 0$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\lambda = 1$ $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\lambda = -1$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
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5 pts each col



$$P = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

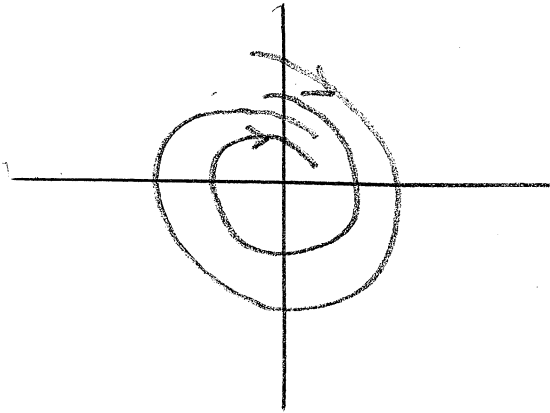
$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2 pts

3 pts
for
normalizing
cols

4. (20) The matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1 + 2i$ with eigenvector $\begin{pmatrix} -1 - 2i \\ 1 \end{pmatrix}$ and $\lambda_2 = -1 - 2i$ with eigenvector $\begin{pmatrix} -1 + 2i \\ 1 \end{pmatrix}$. Determine the type and stability of the origin for the system $dy/dt = Ay$. Find a real general solution, and sketch some trajectories in the phase plane (indicate directions of trajectories).

3 pts
for
CW
in



$$\text{Ckt } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{dy}{dt} = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

5

type

Spiral

or Asymp.
Stable

stability

stable and attractive

5

2

real general solution

$$C_1 e^{-t} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right)$$

$$+ C_2 e^{-t} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right)$$

5 pts

5. (10) For the equation $y'' - y + y^2/2 = 0$, convert to a corresponding system
 $dx_1/dt = f_1(x_1, x_2)$ $dx_2/dt = f_2(x_1, x_2)$.
 (Do not attempt to solve the system.)

$$x_1 = y$$

$$x_2 = y'$$

system
$\frac{dx_1}{dt} = x_2$
$\frac{dx_2}{dt} = x_1 - \frac{x_1^2}{2}$

10 pts

6. (10) Given that $(-2, 2)$ is a critical point of the non-linear system
 $dx_1/dt = -x_1 + x_2 + x_1x_2$ $dx_2/dt = -x_1 - x_2$,
 find the type and stability of the system at $(-2, 2)$.

$$\frac{dx_1}{dt} = -(x_1 - 2) + (x_2 + 2) + (x_1 - 2)(x_2 + 2)$$

$$\frac{dx_2}{dt} = -(x_1 - 2) - (x_2 + 2)$$

linearized system

$$\frac{d\tilde{x}_1}{dt} = \tilde{x}_1 - \tilde{x}_2$$

$$\frac{d\tilde{x}_2}{dt} = -\tilde{x}_1 - \tilde{x}_2$$

5 pts

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-1-\lambda) - 1$$

$$= -2 + \lambda^2$$

$$\lambda = \pm \sqrt{2}$$

type and stability

saddle point

unstable

5 pts