Problem 1: Consider the matrix

$$A = \begin{bmatrix} 8 & 2 & 6 \\ 16 & 6 & 32 \\ 4 & 0 & -7 \end{bmatrix}$$

a) Solve $Ax = 0$ and write down a basis for the null space. *(10 points)*

b) What is the nullity of A? *(3 points)*

c) What is the rank of A? *(3 points)*

d) Does the inverse of A exist? Why or why not? *(4 points)*

Problem 2: Consider the matrix

$$A = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}$$

a) Find the eigenvalues and corresponding eigenvectors of A. *(8 points)*

b) Diagonalize A, that is find P and D such that D is diagonal, and $P^{-1}AP = D$. *(7 points)*

c) Calculate P^{-1}. *(5 points)*

Problem 3: Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

a) Is the matrix A orthogonal? Give a reason for your answer. *(8 points)*

b) What do you know about the eigenvalues of A from you answer in part a)? *(4 points)*

c) Calculate the eigenvalues and eigenvectors of A. *(8 points)*

Problem 4: Consider the second order differential equation

$$y'' + 25y = 0$$

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a) Convert the equation to a system of first–order differential equations. *(5 points)*

b) Calculate the general solution of the system found in part a). In case your answer is complex, write down a real general solution. *(7 points)*

c) What kind of critical point is the origin? Is it stable or unstable? *(4 points)*

d) Draw a qualitative phase portrait for the vicinity of the origin. *(4 points)*

Problem 5: Consider the system of equations

$$\begin{aligned}\frac{dy_1}{dt} &= -y_1 + y_2 + y_1 y_2 \\ \frac{dy_2}{dt} &= -y_1 - y_2\end{aligned}$$

a) Find the critical points of the system. *(5 points)*

b) In the same order as they are listed in part a), tell if the critical points are nodes, saddle points, centers or spiral points. *(8 points)*

c) In the same order as they are listed in part a), tell if the critical points are stable, stable and attractive or unstable. Give a reason for your answer in each case. *(7 points)*