More solutions to practice problems

3. $$\begin{pmatrix} 2 & -1 \\ 8 & -5 \end{pmatrix}^{-1} = \frac{1}{(2)(-5)-(-1)(8)} \begin{pmatrix} -5 & -(-1) \\ -(8) & 2 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} -5 & 1 \\ -8 & 2 \end{pmatrix} = \begin{pmatrix} 5/2 & -1/2 \\ 4 & -1 \end{pmatrix}$$

D.

4. Suppose $A\vec{x} = \vec{b}$ has <u>no</u> sol^{n}s. A $n \times n$.

$$[A \mid \vec{b}] \sim [0 \cdots 0 \mid 1]$$

Rank $A < n$, $\det(A) = 0$

Rows are <u>dependent</u>.

$A\vec{x} = \vec{0}$ $\leftarrow$ free variables arise. ∞ many sol^{n}s.

A non-singular: A^{-1} exists, $\det(A) \neq 0$

i) $A\vec{x} = \vec{0}$ has ∞ many sol^{n}s.

ii) Rank$(A) < n$.

iii) $\mathbb{A}$ has no inverse.

Word: Singular: $\mathbb{A}^{-1}$ does not exist. $\det(\mathbb{A}) = 0$

non-singular: $\mathbb{A}^{-1}$ does exist. $\neq 0$

$$[\mathbb{A} \mid \mathbb{I}] \rightsquigarrow [\mathbb{I} \mid \mathbb{A}^{-1}]$$ ← Jordan method

$$\begin{pmatrix} 2 & -1 \\ 8 & -5 \end{pmatrix}^{-1} = \frac{1}{(-2)} \begin{pmatrix} -5 & 1 \\ -8 & 2 \end{pmatrix}$$

Are $\vec{v}_1, \ldots, \vec{v}_m$ dependent?

$\boxed{\dim(\text{Row Space}) = \dim(\text{Col Space})}$

Put $\vec{v}$'s in as the rows of a matrix. Do row operations. Row of zeroes on bottom: depen.

Square: $\det(\mathbb{A}) = 0$ means dep.

8. $\vec{y}\,' = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \vec{y}$ $\qquad \vec{y} = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$ is a sol^n.

another <u>lin ind</u> sol^n is.

e. vals $r = 2, 3$. $\qquad r = 2: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$

$\det\begin{pmatrix} 3-r & 0 \\ 1 & 2-r \end{pmatrix} = 0 \qquad r = 3: \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$

$\vec{y}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \qquad \vec{y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$

$$\begin{pmatrix} e^{3t} \\ e^{2t} + e^{3t} \end{pmatrix} = \vec{y}_1 + \vec{y}_2$$ ← this is a sol^n by superpostion

Lin. Indep? yes $c_1\vec{y}_1 + c_2(\vec{y}_1+\vec{y}_2) = 0$

$(c_1+c_2)\vec{y}_1 + c_2\vec{y}_2 = 0$

lin indep.

(1) $c_1+c_2=0$

(2) $c_2=0$

must

$c_1=0,\ c_2=0$

9. $\begin{cases} y_1' = y_1 + 3y_2 \\ y_2' = 4y_1 + 2y_2 \end{cases}$ $\quad \vec{y}\,' = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\vec{y}.$

$\mathrm{Det}\begin{pmatrix} 1-r & 3 \\ 4 & 2-r \end{pmatrix} = (1-r)(2-r)-12$

$r^2-3r-10=0$

$(\mathbb{A}-r\mathbb{I})\vec{a}=\vec{0}$

$r=\dfrac{3\pm\sqrt{9+40}}{2} = \dfrac{3\pm7}{2} = 5,\ -2$

Saddle Point.

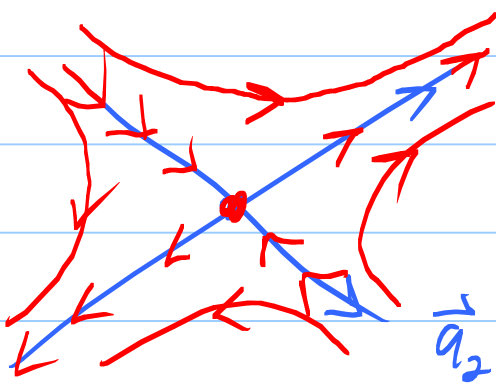
Unstable

$\underline{r=5}:\ \left(\begin{array}{cc|c} 1-5 & 3 & 0 \\ 4 & 2-5 & 0 \end{array}\right)$

$\begin{array}{cc} A & B \end{array}$

$\left(\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \leftarrow$ know

See $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad \begin{pmatrix} -B \\ A \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix}$



$\vec{a}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad r=5$

$\vec{a}_2 \quad r=-2$

$\underline{r=5}:\ (\mathbb{A}-r\mathbb{I})\vec{a}=\vec{0}$

$\uparrow r=5$

$\left(\begin{array}{cc|c} 1-5 & 3 & 0 \\ 4 & 2-5 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\vec{a}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} B \\ -A \end{pmatrix} \text{ or } \begin{pmatrix} -B \\ A \end{pmatrix}$$

12. $$\begin{cases} \dfrac{dx}{dt} = y = f(x,y) & f(0,0) = 0 \checkmark \\ \dfrac{dy}{dt} = \sin x = g(x,y) & g(0,0) = \sin 0 = 0 \checkmark \end{cases}$$

Linearize: $$\mathbb{A} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x_0, y_0)}$$

$\vec{x}' = \mathbb{A}\vec{x}$

$$= \begin{bmatrix} 0 & 1 \\ \cos x & 0 \end{bmatrix}\Bigg|_{(0,0)} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbb{A}}$$

Linearized system at $(0,0)$ is $\vec{x}' = \mathbb{A}\vec{x}$

where $\mathbb{A} = \mathbb{J}\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ \cos 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det(\mathbb{A} - r\mathbb{I}) = \det\begin{pmatrix} 0-r & 1 \\ 1 & 0-r \end{pmatrix} = r^2 - 1 = 0.$$

$r = \pm 1.$ ← unstable saddle point.

11. Plug in $(1,1)$ in J. Ugly!

13. Given: $$\mathcal{L}\Big(\underbrace{\frac{e^{-1/(4t)}}{\sqrt{t}}}_{f(t)}\Big) = \underbrace{\frac{\sqrt{\pi}\, e^{-\sqrt{s}}}{\sqrt{s}}}_{F(s)}.$$

Find: $$\mathcal{L}\left(\frac{e^{-1/(4t)}}{t^{3/2}}\right) = \mathcal{L}\left(\frac{f(t)}{t}\right)$$

$$= \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{\sqrt{\pi}\, e^{-\sqrt{\sigma}}}{\sqrt{\sigma}}\,d\sigma$$

Let $u = \sqrt{\sigma} = \sigma^{1/2}$ $\quad du = \frac{1}{2}\sigma^{-1/2} = \frac{1}{2\sqrt{\sigma}}$

When $\sigma = s$, $u = s^{1/2} = \sqrt{s}$

$$2\sqrt{\pi}\int_s^\infty e^{-\sqrt{\sigma}}\,\frac{1}{2\sqrt{\sigma}}\,d\sigma = 2\sqrt{\pi}\int_{u=\sqrt{s}}^\infty e^{-u}\,du$$

$$= 2\sqrt{\pi}\lim_{u\to\infty}\left[-e^{-u} + e^{-\sqrt{s}}\right] = \underline{\underline{2\sqrt{\pi}\, e^{-\sqrt{s}}}}$$

22. Given $\mathcal{F}\left(e^{-x^2/2}\right) = e^{-\omega^2/2}$.

Find $\mathcal{F}\left(x e^{-x^2/2}\right)$

$f'(x) = -x e^{-x^2/2}$. Aha!

$$\boxed{\tilde{\mathcal{F}}(f') = i w \hat{f}(w)}$$

$$\tilde{\mathcal{F}}\left(-x e^{-x^2/2}\right) = i w e^{-w^2/2}$$

$$\tilde{\mathcal{F}}\left(x e^{-x^2/2}\right) = -i w e^{-w^2/2}$$

28. $\Delta u = 0 \quad r < 1$

$u(1,\theta) = \cos 2\theta$

D. Prob. on disc.

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left(a_n \left(\frac{r}{R}\right)^n \cos n\theta + b_n \left(\frac{r}{R}\right)^n \sin n\theta\right)$$

$r = R$: $u(R,\theta) =$ Fourier Series.

$R = 1$. $u(r,\theta) = r^2 \cos 2\theta$

$$u\left(\tfrac{1}{2}, \tfrac{\pi}{4}\right) = \left(\tfrac{1}{2}\right)^2 \cos 2\left(\tfrac{\pi}{4}\right) = 0$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix}$$

Dim of col space? = 2 ← dim(Row Space)

Note: Col Space $\subset \mathbb{R}^2$ and has dim 2.

It must $= \mathbb{R}^2$

Basis: $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$