NAME ______________________

YOU MUST SHOW WORK ON ALL PROBLEMS TO RECEIVE CREDIT.

1. (15) Find $\mathcal{L}^{-1}(\frac{se^{-2s}}{(s+3)^2+9})$.

$$\mathcal{L}^{-1}\left(\left(\frac{s+3}{(s+3)+9} - \frac{3}{(s+3)^2+9}\right)e^{-2s}\right)$$

$$= u(t-2)\left(e^{-3(t-2)}\cos(3(t-2)) - e^{-3(t-2)}\sin(3(t-2))\right)$$

2. (15) Solve by Laplace transforms $y'' + 5y' + 6y = 3\delta(t-2)$, $y(0) = y'(0) = 0$.

$$s^2 Y + 5sY + 6Y = 3e^{-2s}$$

$$Y = \frac{3e^{-2s}}{s^2+5s+6} = \frac{3e^{-2s}}{(s+3)(s+2)}$$

$$\frac{3}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} \Rightarrow A(s+2) + B(s+3) = 3$$

or $A + B = 0$

$2A + 3B = 3$

$\Rightarrow A = -3 \quad B = 3$

$$\mathcal{L}^{-1}\left\{ \frac{-3e^{-2s}}{s+3} + \frac{3e^{-2s}}{s+2} \right\}$$

$$= 3u(t-2)\left(e^{-2(t-2)} - e^{-3(t-2)}\right)$$

3. For the Sturm-Liouville problem

$$y'' + \lambda y = 0 \quad y(0) = 0,\ y'(1) = 0,$$

(10) (a) find the eigenvalues. General Solution:

① $\lambda < 0$ $\quad c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x}$

② $\lambda = 0$ $\quad c_1 + c_2 x$

③ $\lambda > 0$ $\quad c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$

① $c_1 + c_2 = 0 \quad c_1 e^{\sqrt{|\lambda|}} - c_2 e^{-\sqrt{\lambda}} = 0 \Rightarrow c_1 (e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) = 0$

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

② $c_1 = 0, \quad c_2 = 0$

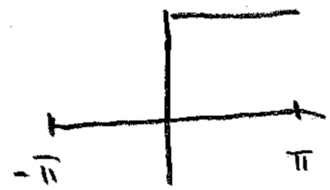
③ $c_1 = 0, \quad c_2 \sqrt{\lambda} \cos\sqrt{\lambda} = 0 \Rightarrow \lambda = \left(\frac{\pi}{2} + n\pi\right)^2 \quad n = 0, 1, \ldots$

(10) (b) In the same order as the eigenvalues are listed above, give corresponding eigenfunctions.

$\sin\left(\frac{\pi}{2} + n\pi\right)x \qquad n = 0, 1, 2, \ldots$

4. (20) Let $f(x)$ be a 2π periodic function such that

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$



Find the Fourier series for $f(x)$ both in the real trigonometric form and complex form. (The next page is blank in case you need extra paper to show work.)

$a_0 = \frac{1}{2\pi}\int_0^\pi 1\,dx = \frac{1}{2}$ $\quad n>0 \quad a_n = \frac{1}{\pi}\int_0^\pi \cos nx\,dx = \frac{1}{\pi n}\sin nx\Big|_0^\pi = 0$

$b_n = \frac{1}{\pi}\int_0^\pi \sin nx\,dx = -\frac{1}{n\pi}\cos nx\Big|_0^\pi = \frac{1}{n\pi}\left(1-\cos n\pi\right) = \frac{1-(-1)^n}{n\pi}$

$c_0 = a_0$

$c_n = \frac{a_n - ib_n}{2} = -i\,\frac{1+(-1)^n}{2n\pi}$

$c_{-n} = \frac{a_n + ib_n}{2} = i\,\frac{1+(-1)^n}{2n\pi}$

real form

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n}\sin nx}$$

complex form

$$\boxed{f(x) = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(1-(-1)^n)i}{2\pi n} e^{inx}}$$

5. (20) Let

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier cosine transform of f, and the complex Fourier transform of f. (The next page is blank in case you need extra paper to show work.)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^1 x \cos wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{x}{w} \sin w \Big|_0^1 - \int_0^1 \frac{\sin wx}{w} dx \right)$$

$u = x \quad dv = \cos wx\, dx$

$du = dx \quad v = \frac{1}{w} \sin wx$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{1}{w^2} \right)$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^1 x e^{-iwx} dx$$

$u = x \quad dv = e^{-iwx} dx$

$du = dx \quad v = \frac{i}{w} e^{-iwx}$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{xi}{w} e^{-iwx} \Big|_0^1 - \frac{i}{w} \int_0^1 e^{-iwx} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{ie^{-iw}}{w} + \frac{1}{w^2} e^{-iwx} \Big|_0^1 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{ie^{-iw}}{w} + \frac{e^{-iw}}{w^2} - \frac{1}{w^2} \right)$$

Fourier cosine transform

$$\boxed{\sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{1}{w^2} \right)}$$

complex Fourier transform

$$\boxed{\frac{1}{\sqrt{2\pi}} \left(\frac{ie^{-iw}}{w} + \frac{e^{-iw}}{w^2} - \frac{1}{w^2} \right)}$$

6. (10) Given that the Fourier series for $f(x) = x/2$ for $-\pi < x < \pi$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \text{ compute the sum } \sum_{n=1}^{\infty} \frac{1}{n^2},$$

$$\int_{-\pi}^{\pi} \frac{x^2}{4}\,dx = \frac{1}{12} x^3 \Big|_{-\pi}^{\pi} = \frac{\pi^3}{6}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$