Solutions to selected practice problems.

1. Subspace Test: 1) If v_1 and v_2 in S', then v_1+v_2 must be in S'. 2) If v is in S', then cv must be too.

   i) $\det \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{v_1} = 0$, $\det \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{v_2} = 0$,

   but $\det(v_1+v_2) = 1 \neq 0$.

   ii) If $A_1 B = BA_1$ and $A_2 B = BA_2$, then

   $$A_1B + A_2B = BA_1 + BA_2$$
   $$(A_1+A_2)B = B(A_1+A_2).$$ So A_1+A_2 is in.

   Also, if $AB = BA$, then

   $$c(AB) = c(BA)$$
   $$=(cA)B = B(cA)$$ So cA is in.

2. i) The 3rd vector is 3 times second minus the first, so the[y] are dependent.

   ii) Calculate $\det(A)$. Get zero. They are dependent.

   iii) Row of zeroes means $\det(A)=0$. They are dependent.

3., 4. See lectures.

5. Easy.

6. Eigenvalues are roots of $\det\begin{pmatrix} 1-r & 0 & 1 \\ 0 & 2-r & 0 \\ 3 & 6 & 3-r \end{pmatrix} =$

$$(+1)(2-r)\det\begin{bmatrix} 1-r & 1 \\ 3 & 3-r \end{bmatrix} = (2-r)\left[(1-r)(3-r)-3\right]$$

$$= (2-r)\left(r^2-4r\right) = r(2-r)(r-4). \quad r = 0, 2, 4.$$

7. For $r=-1$: $\begin{pmatrix} 0-(-1) & -1 & 0 \\ -1 & 1-(-1) & -1 \\ 0 & -1 & 0-(-1) \end{pmatrix} \vec{a} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a_1 = a_3 = s \\ \leftarrow a_2 = a_3 = s \end{array}$$

free

$a_3 = s$

$\vec{a} = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a basis for the eigenspace for $r=-1$.

8. See lecture notes.

9-12. See notes.

14. $$\frac{4}{s^3+4s} = \frac{4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$4 = A(s^2+4) + (Bs+C)s$$

$$4 = \underbrace{(A+B)}_{0}s^2 + \underbrace{C}_{0}s + 4\underbrace{A}_{1} \qquad \boxed{\begin{array}{l} A=1 \\ B=-1 \\ C=0 \end{array}}$$

So $\frac{4}{s^3+4s} = \frac{1}{s} - \frac{s}{s^2+2^2}$ and $\mathcal{L}^{-1} = 1 - \cos 2t$.

15. $u(t-1)f(t-1)$ where $f(t) = e^{2t}$.

18. The given function is $\frac{1}{2}$ + (ODD function). Only one graph is odd after being slid down by any amount. (C)

19. The given function is even. Only (A) is even.

20. $$c_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i\cdot 1\cdot x}\,dx = \frac{1}{2\pi}\int_{\pi/2}^{\pi} e^{-ix}\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{1}{i}e^{-ix}\right]_{\pi/2}^{\pi} = \frac{1}{2\pi}\left[-\frac{1}{i}\left(e^{-i\pi} - e^{-i\pi/2}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{i}\left((-1) - (-i)\right)\right] = \frac{1}{2\pi}\left(i(-1+i)\right) = \frac{-i-1}{2\pi}$$

21. The key to this problem is that

$$\tilde{\mathcal{F}}_c\left(\tilde{\mathcal{F}}_c(f(x))\right) = f(x).$$

23. We know via separation of variables that

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin nx \cos nt$$ where the

a_n are the Fourier Sine Coefficients for $f(x)$.

So $u(x,t) = \frac{1}{2}\sin 2x \cos 2t + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{4-(2n+1)^2}\sin(2n+1)x \cos(2n+1)t$.

$$u\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \frac{1}{2}\sin\frac{2\cdot\pi}{4}\cos\frac{2\cdot\pi}{2} + 0$$

since $\cos(\text{odd})\frac{\pi}{2} = 0$.

24. Use D'Alembert's sol^n: $u(x,t) = \varphi(x-2t) + \psi(x+2t)$.

Then $\frac{\partial u}{\partial t} = -2\varphi'(x-2t) + 2\psi'(x+2t)$ and we need

want

$$\begin{cases} u(x,0) = \varphi(x) + \psi(x) = \sin x & (A) \\ u_t(x,0) = -2\varphi'(x) + 2\psi'(x) = \sin 2x & (B) \end{cases}$$

Integrate (B): $-2\varphi(x) + 2\psi(x) = -\frac{1}{2}\cos 2x$

$$\begin{cases} \varphi(x) + \psi(x) = \sin x & (A) \\ -2\varphi(x) + 2\psi(x) = -\frac{1}{2}\cos 2x & (B)' \end{cases}$$

Cramer's
$$\varphi(x) = \frac{\det\begin{bmatrix} \sin x & 1 \\ -\frac{1}{2}\cos 2x & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}} = \frac{1}{2}\sin x + \frac{1}{8}\cos 2x$$

$$\psi(x) = \frac{\det\begin{bmatrix} 1 & \sin x \\ -2 & -\frac{1}{2}\cos 2x \end{bmatrix}}{4} = \frac{1}{2}\sin x - \frac{1}{8}\cos 2x$$

Finally, $u\left(\frac{\pi}{4}, \frac{\pi}{8}\right) = \varphi\left(\frac{\pi}{4} - 2\cdot\frac{\pi}{8}\right) + \psi\left(\frac{\pi}{4} + 2\cdot\frac{\pi}{8}\right)$

$$= \varphi(0) + \psi\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{1}{2}\sin 0 + \frac{1}{8}\cos 0\right) + \left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{8}\cos \pi\right)$$

$$= \frac{1}{8} + \frac{1}{2} - \frac{1}{8}(-1) = \frac{6}{8} = \frac{3}{4}.$$

25. We know that
$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\sin\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

where a_n are the Fourier Sine Coefficients for f.
Here, $L=\pi$, $c=2$. So
$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3}(\sin nx)\, e^{-(2n)^2 t}$$

26. Use D'Alembert's sol^n: $u(x,t) = \varphi(x-2t) + \psi(x+2t)$

Very similar to 24. Get

$$\begin{cases} \varphi(x) + \psi(x) = \sin x \\ -2\varphi(x) + 2\psi(x) = \cos x \end{cases}$$

$\varphi(x) = \frac{1}{4} \sin x$, $\psi(x) = \frac{3}{4} \sin x$

$$u(x,t) = \frac{1}{4} \sin(x-2t) + \frac{3}{4} \sin(x+2t)$$

$$u(0, \tfrac{\pi}{4}) = \frac{1}{4} \sin(-\tfrac{\pi}{2}) + \frac{3}{4} \sin(\tfrac{\pi}{2}) = -\frac{1}{4} + \frac{3}{4} = 1/2.$$

27. Problem 1 on p. 568 with $a=1$, $c=2$.

$$u(x,t) = \int_0^\infty [A(p) \cos px + B(p) \sin px] e^{-c^2p^2t} \, dp$$

where $B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underbrace{f(v)}_{\text{even}} \underbrace{\sin pv}_{\text{odd}}}_{\text{odd}} \, dv = 0$ and

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underbrace{f(v)}_{\text{even}} \underbrace{\cos pv}_{\text{even}}}_{\text{even}} \, dv = \frac{2}{\pi} \int_0^\infty f(v) \cos pv \, dv$$

$$= \frac{2}{\pi} \int_0^1 \cos pv \, dv = \frac{2}{\pi} \left[\frac{1}{p} \sin pv \right]_{v=0}^{1} = \frac{2}{\pi} \frac{\sin p}{p}.$$

So $u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\sin p}{p} \cos px \, e^{-4p^2t} \, dp.$

28. $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right)$

where a's, b's are Fourier Coeff for $f(\theta) = \cos 2\theta$.
But $\cos 2\theta$ is its own Fourier Series. So

$$u(r,\theta) = r^2 \cos 2\theta.$$

29. Separation of variables (see p. 557 plus lecture notes on 12.5) yields

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \, e^{-\left(\frac{cn\pi}{L}\right)^2 t} \qquad \begin{pmatrix} c=2 \\ L=\pi \end{pmatrix}$$

where A_n are the Fourier Cosine Coeff. for $f(x)$.

$$\text{So} \quad u(x,t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \left(\cos 2nx\right) e^{-(2(2n))^2 t}$$

match

30. $$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty} \frac{2^2(\pm 1)^2}{n^2} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{1}{3}\pi^3 - 0\right] = \frac{2\pi^2}{3}.$$

$$\text{So} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$