Math 530

Homework 1

1. a) If a is a complex number in the unit disc, define

$$\phi_a(z) = \frac{z-a}{1-\bar{a}\,z}.$$

Prove that $|\phi_a(z)| = 1$ if $|z| = 1$.

b) Prove that $|\phi_a(z)| < 1$ if $|z| < 1$.

c) Write out $w = \phi_a(z)$ and solve for z to get a formula for the inverse mapping to ϕ_a. Show that $\phi_a(z)$ is a *one-to-one* analytic mapping of the open unit disk *onto* itself as a function of z.

2. Prove that $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

What does this identity say about parallelograms?

3. Suppose that f is an analytic function on an open set Ω_1 which maps into an open set Ω_2 on which g is defined and analytic. Prove the complex chain rule, which says that $h(z) = g(f(z))$ is analytic on Ω_1 and that $h'(z) = g'(f(z))f'(z)$.

4. Suppose $z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are continuously differentiable real functions on the interval $[a,b]$. Write $z'(t) = x'(t) + iy'(t)$. Show that if f is analytic on $\mathbb{C}$, then $w(t) = f(z(t))$ is such that $w'(t) = f'(z(t))z'(t)$ on $[a,b]$. This is another important Chain Rule.

5. Show that a sequence of complex numbers $\{a_n\}$ converges to b if and only if Re $a_n \to$ Re b and Im $a_n \to$ Im b. Also, show that $\{a_n\}$ is a Cauchy sequence if and only if Re a_n and Im a_n are Cauchy. Conclude that the completeness of the complex number system follows from the completeness of the reals.

6. Prove that an absolutely convergent series of complex numbers is convergent.

7. Show that the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$ is given by the supremum of the set of real numbers $r \geq 0$ with the property that the sequence $|a_n|r^n$ is a bounded sequence.