Math 530

Homework 2

1. Suppose that $\varphi(z)$ is a continuous function on the trace of a path γ. Prove that the function

$$f(z) = \int_\gamma \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta$$

is analytic on $\mathbb{C} - \operatorname{tr}\gamma$.

2. Suppose that a_n is a sequence of non-zero complex numbers. Show that if

$$R = \lim_{n\to\infty} |a_n|/|a_{n+1}|$$

exists, then R is equal to the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

3. Suppose $\{a_n\}_{n=1}^{\infty}$ is an enumeration of the set of complex numbers in the unit disc with rational real and imaginary parts. Prove that the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ is equal to one.

4. Show that

$$\sum_{n=-N}^{N} e^{in\theta} = 1 + 2\sum_{n=1}^{N} \cos n\theta = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}.$$

Hint: Use $\sum_{n=-N}^{N} w^n = 1 + w^{-1}\sum_{n=0}^{N-1}(w^{-1})^n + w\sum_{n=0}^{N-1} w^n$ and the formula for the sum of a geometric series.

5. Suppose that an analytic function is written in polar form

$$f(re^{i\theta}) = U(r,\theta) + iV(r,\theta).$$

Derive the polar form of the Cauchy-Riemann equations,

$$rU_r = V_\theta \qquad \text{and} \qquad U_\theta = -rV_r.$$

Prove that if U and V are continuously differentiable and satisfy the polar Cauchy-Riemann equations on some polar rectangle, then $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ defines an analytic function there. Use this result to verify that the function $\operatorname{Log}(re^{i\theta}) = \operatorname{Ln} r + i\theta$ is analytic on $\{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$. Explain why $\ln|z|$ and $\operatorname{Arg} z$ are harmonic functions on this set.