Math 530

Homework 5

1. Prove that there is no analytic function f on the unit disk such that $f(1/n) = (-1)^n/n$ for $n = 2, 3, 4, \dots$

2. If $f(z)$ is analytic on a domain Ω, show that $\overline{f(\bar{z})}$ is analytic on $\{z : \bar{z} \in \Omega\}$.

3. Suppose that Ω is a domain in $\mathbb{C}$ that is symmetric with respect to the real axis. If $f(z)$ is an analytic function on Ω that is real-valued on a non-empty open interval of the real line contained in $\Omega \cap \mathbb{R}$, prove that $f(\bar{z}) = \overline{f(z)}$ for all z in Ω.

4. Suppose that F is a *one-to-one* analytic mapping of a domain Ω *onto* the unit disc such that $F(a) = 0$. Prove that if g is any analytic function on Ω which maps Ω into the unit disc such that $g(a) = 0$, then $|g'(a)| \leq |F'(a)|$. If $|g'(a)| = |F'(a)|$, does it follow that $g \equiv F$? (You will need to use the fact that if $F : \Omega_1 \to \Omega_2$ is a one-to-one analytic mapping of a domain Ω_1 onto a domain Ω_2, then the inverse mapping F^{-1} is analytic on Ω_2.)

5. Suppose that F is a *one-to-one* analytic mapping of the unit disc *onto* a domain Ω. Show that if g is any other analytic map of the unit disc into Ω such that $g(0) = F(0)$, then $g(D_r(0)) \subset F(D_r(0))$ for all $0 < r < 1$.

6. Suppose that F is a *one-to-one* analytic mapping of the unit disc onto a square with center at the origin. Prove that, if $F(0) = 0$, then $F(iz) = iF(z)$ for all z.

7. Suppose that f is an analytic function on a domain Ω such that for each point $a \in \Omega$, there is some coefficient c_n which is zero in the power series expansion $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$ at a. Prove that f must be a polynomial. (Note that n may depend on a.) *Hint:* Let $\mathcal{O}_n$ denote the set consisting of points $z \in \Omega$ such that $f^{(n)}(z) = 0$. Notice that $\Omega = \cup_{n=0}^{\infty} \mathcal{O}_n$.

8. Prove that if G_1 and G_2 are two analytic functions on a domain Ω such that $e^{G_1} \equiv e^{G_2}$ on Ω, then they differ by an integer multiple of $2\pi i$. What can you say about two analytic functions with the same N-th power?

9. Prove that the ring of analytic functions on a domain is an integral domain.