Math 530

Homework 7

1. Find a one-to-one conformal mapping of the region common to the two disks $|z-1| < \sqrt{2}$ and $|z+1| < \sqrt{2}$ onto the unit disk.

2. Find a one-to-one conformal mapping of the region $\{z : 0 < \text{Re } z < 1\}$ onto the unit disk. Use the inverse of this map to show that there exists a bounded harmonic function on the unit disk whose harmonic conjugates are unbounded on the unit disk. (Remark: A similar idea can be used to show that there is a harmonic function which extends continuously to the closed disk that does not have a bounded harmonic conjugate on the disk.)

3. Let Ω denote the open set obtained by removing the interval $[-1, 1]$ from $\mathbb{C}$. Prove that there is an analytic function $F(z)$ on Ω such that $F(z)^2 = \frac{z+1}{z-1}$. *Hint:* What is the image of Ω under the map $(z+1)/(z-1)$?

4. Suppose that f_n is a sequence of analytic functions on a domain Ω which converges uniformly on compact subsets of Ω to a non-constant function f. Suppose that f has a zero of order m at a point a in Ω. Prove that there is an $\epsilon > 0$ and a positive integer N such that each function $f_n(z)$ with $n > N$ has exactly m zeroes (counted with multiplicity) on $D_\epsilon(a) \subset \Omega$.

5. Suppose that f_n is a sequence of analytic functions on a domain Ω which converges uniformly on compact subsets of Ω to a function f. Suppose that $\widetilde{\Omega}$ is a domain containing $f_n(\Omega)$ for each n. Prove that, if f is not constant, then $\widetilde{\Omega}$ contains $f(\Omega)$ too.

6. Let $\mathcal{F}$ denote the set of all analytic functions f that map the the upper half plane into the unit disc. Let $M = \sup\{|f'(i)| : f \in \mathcal{F}\}$. Show that $M < \infty$. Find all functions, if any, in $\mathcal{F}$ such that $|f'(i)| = M$.

7. Find a conformal mapping of the upper half plane minus the closed line segment joining the origin to the point i that is one-to-one and onto the unit disc.