Lecture 2 The Cauchy-Riemann equations (C-R eqns)

Topology of $\mathbb{C} \approx \mathbb{R}^2$ topology

$D_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| < r \}$ ← open disc of radius r about z_0

$\Omega \subset \mathbb{C}$ is <u>open</u> if, for each $z_0 \in \Omega$, there is an $r > 0$ such that $D_r(z_0) \subset \Omega$.

<u>Def^n</u> $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$: Given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ when $|z - z_0| < \delta$, $z \neq z_0$.

Look at <u>real</u> calc book. Change x's to z's.

$|x| \sim |z|$ ← modulus

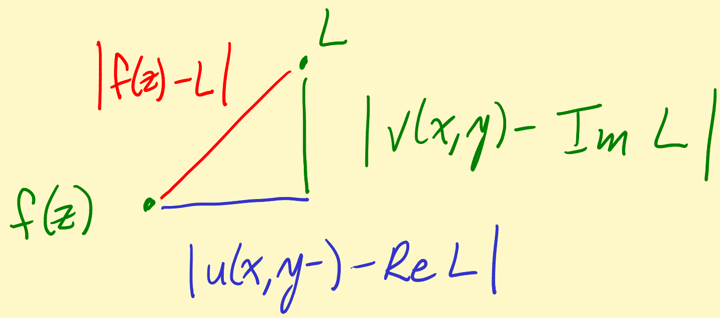
<u>Fact</u>: Basic rules and facts of $\mathbb{R}$ calculus $\rightsquigarrow$ $\mathbb{C}$ calc.

<u>EX</u>: $(fg)' = f'g + fg'$

Chain rule too!

<u>Little lemma</u>: $f(x+iy) = u(x,y) + i\,v(x,y)$, $z = x+iy$, $z_0 = x_0 + iy_0$

$$\lim_{z \to z_0} f(z) = L \iff \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = \operatorname{Re} L \\ \lim_{(x,y) \to (x_0,y_0)} v(x,y) = \operatorname{Im} L \end{cases}$$



Def^n: f is complex diff'ble at $a \in \mathbb{C}$:

$$\underbrace{\lim_{z \to a} \frac{f(z)-f(a)}{z-a}}_{DQ} \quad \text{exists. Call it } f'(a).$$

MA 530 : Complex fcns that are $\mathbb{C}$-diff'ble on an open set Ω. $\leftarrow$ analytic fcns. (or holomorphic)

EX: $f(z) = z^2$

$$DQ = \frac{z^2 - a^2}{z-a} = z + a \xrightarrow[\text{as } z \to a.]{} a + a = 2a$$

EX: $f(z) = \bar{z}$ $\quad z = x+iy$. $\quad \bar{z} = x - iy$ (conjugate)

is not $\mathbb{C}$-diff.

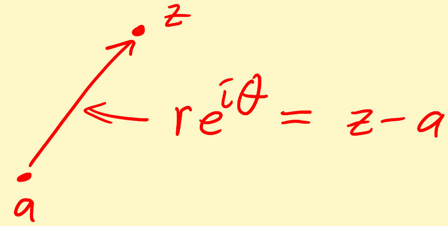
Facts $\quad |zw| = |z||w| \qquad \overline{(zw)} = \bar{z}\,\bar{w}$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, $w \neq 0$. $\qquad \overline{(z/w)} = \bar{z}/\bar{w}$

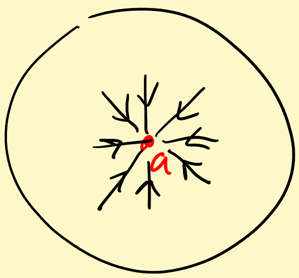
$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \overline{(e^{i\theta})} = \frac{1}{e^{i\theta}}$ $\quad$ ($N=-1$ DeMoivre's)

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$DQ = \frac{\bar{z}-\bar{a}}{z-a}$$



$$= \frac{\overline{(re^{i\theta})}}{re^{i\theta}} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-i\theta}\underbrace{\left(\frac{1}{e^{i\theta}}\right)}_{e^{-i\theta}} = e^{-2i\theta}$$



Different limits along rays!

<u>Thm</u> f $\mathbb{C}$-diff'ble at $a \Rightarrow f$ cont at a.

<u>Pf</u>: $$\frac{f(z)-f(a)}{z-a} = f'(a) + E(z)$$ where $E(z) \to 0$ as $z \to a$.

$$f(z) = \underbrace{f(a) + f'(a)(z-a)}_{\text{complex linear}} + E(z)\cdot(z-a)$$

$z \to a$

$$f(a) + f'(a)\cdot 0 + 0\cdot 0$$

So $f(z) \to f(a)$ as $z \to a$.

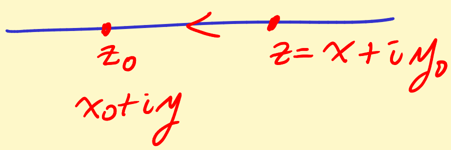
<u>Cauchy-Riemann eqns</u> If $f(x+iy) = u(x,y) + iv(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0 + iy_0$, then the first partial derivatives of u and v <u>exist</u> at (x_0,y_0) and

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{at } (x_0, y_0)$$

Notation $\quad u_x = \dfrac{\partial u}{\partial x}$ etc. $\qquad u^0 = u(x_0, y_0)$

$v_x^0 = v_x(x_0, y_0)$ etc.

Pf: Step 1: $z \to z_0$ along horiz



$$DQ = \frac{[u(x,y_0) - u(x_0,y_0)] + i\,[v(x,y_0) - v(x_0,y_0)]}{(x+iy_0) - (x_0+iy_0)}$$

$$= \frac{u(x,y_0) - u^0}{x - x_0} + i\,\frac{v(x,y_0) - v^0}{x - x_0}$$

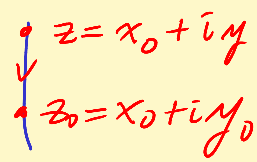
Baby lemma $\qquad \downarrow \text{Re } f'(z_0) \qquad \downarrow \text{Im } f'(z_0)$

Conclude first partials exist and

$$\boxed{f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)}$$

Red box #1

Step 2. Slide along vertical



$$DQ = \frac{\ell\ell\ell}{(x_0+iy)-(x_0+iy_0)} = \frac{\ell\ell\ell}{i(y-y_0)}$$

$$= \underbrace{\frac{1}{i}}_{-i}\left[\frac{u(x_0,y)-u^0}{y-y_0} + i\,\frac{v(x_0,y)-v^0}{y-y_0}\right]$$

$$= v_y^0 + \left(-i\,u_y^0\right)$$

$$\boxed{f'(z_0) = v_y^0 - i\,u_y^0}$$ Red box #2

Equate red boxes. Get C-R eqns.

EX: $\bar{z} = x - iy$ $\quad \left.\begin{array}{l} u(x,y) = x \\ v(x,y) = -y \end{array}\right\}$ C-R eqns fail!

$\bar{z}$ not $\mathbb{C}$-diff at any pt.

EX: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$ C-R eqns hold!

Is $E(z)$ analytic? Yes!

Partial converse to C-R eqns (Ω open set)

If u, v are $\mathcal{C}^1$-smooth on Ω and satisfy the C-R eqns, then $f(x+iy) = u(x,y) + iv(x,y)$ is analytic on Ω.

Calculus lemma Suppose u is $\mathcal{C}^1$-smooth (u cont, first partials exist and are cont), then

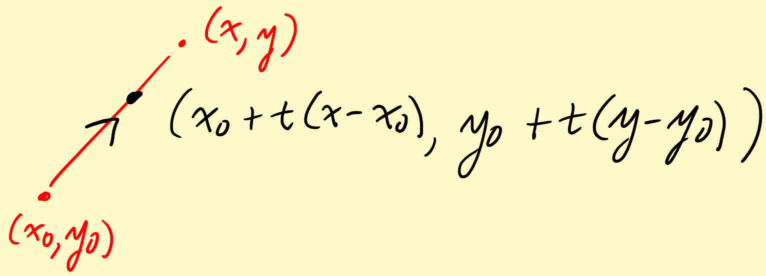
$$u(x,y) = u^0 + u_x^0 \cdot (x-x_0) + u_y^0 \cdot (y-y_0) + R(x,y)$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Pf:

$$u(x,y) - u^0 = \int_0^1 \frac{\partial}{\partial t} u\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big)\, dt$$



$$= \int_0^1 u_x(\quad)\cdot(x-x_0) + u_y(\quad)\cdot(y-y_0)\, dt$$

Now subtract from both sides

$$\underbrace{u_x^0\cdot(x-x_0) + u_y^0\cdot(y-y_0)}_{A}$$

$A = \int_0^1 A\, dt$

$$u - \left[\text{first order Taylor}\right] =$$

$$\int_0^1 \underbrace{\left[u_x(\quad) - u_x^0\right]}_{\text{small}} \cdot (x - x_0) + \underbrace{\left[u_y(\quad) - u_y^0\right]}_{\text{small}} \cdot (y - y_0)\, dt$$

See link on home page for details.