Lecture 3 Cauchy-Riemann eqns, converse. $E(z)=e^z$ properties

Looman-Menchoff Thm Ω open set. u,v real fcns on Ω with

1) u,v continuous on Ω
2) First partials u,v exist and
3) satisfy C-R eqns.

Then $f(x+iy)=u(x,y)+iv(x,y)$ analytic on Ω.

Pf: See R. Narasimhan, Complex analysis in 1 variable, p. 43-50.

Today: Assume u,v $\mathcal{C}^1$-smooth and satisfy C-R eqns.
Then $u+iv$ analytic.

Pf: Need lemma: $u(x,y)=u^0+u^0_x\cdot(x-x_0)+u^0_y\cdot(y-y_0)+R_u(x,y)$

where $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{R(x,y)}{|z-z_0|}\to 0$ as $z=x+iy \to z_0=x_0+iy_0$

$$DQ=\frac{f(z)-f(z_0)}{z-z_0}=\frac{(u+iv)-(u^0+iv^0)}{z-z_0}$$

$$=\frac{[u^0_x\cdot(x-x_0)+\overset{-v^0_x}{u^0_y}\cdot(y-y_0)]+i[v^0_x\cdot(x-x_0)+\overset{u^0_x}{v^0_y}\cdot(y-y_0)]}{z-z_0}+\frac{R_u+iR_v}{|z-z_0|}$$

$$=\frac{[u^0_x+iv^0_x]\cdot[(x-x_0)+i(y-y_0)]}{z-z_0}+\mathcal{E}$$

$$= \underbrace{u_x^0 + i v_x^0}_{\substack{\text{red box \#1} \\ = f'(z_0)}} + \mathcal{E}$$

$$|\mathcal{E}| \le \frac{|R_u|}{|z-z_0|} + \frac{|R_v|}{|z-z_0|} \longrightarrow 0 \text{ as } z \to z_0.$$

Cor: $E(z) = E(x+iy) = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$ is analytic on $\mathbb{C}$.

C-R eqns ✓

E is an entire fcn.

Properties of $E(z)$

0) $E(z)$ is entire ✓

1) $E(0) = 1$ ✓

2) $E'(z) = \begin{cases} u_x + i v_x & \#1 \\ v_y - i u_y & \#2 \end{cases} = E(z)$

3) $E(z+w) = E(z)\,E(w)$

Group homomorphism: Additive Grp $\mathbb{C} \longrightarrow$ Multiplicative Grp $\mathbb{C} - \{0\}$

3) $\Rightarrow$ 4) $E(z)E(-z) = 1$.

$\Rightarrow$ 5) $E(z)$ is never zero.

6) $E(z-w) = E(z)/E(w)$

7) $E(x) = e^x$, $x \in \mathbb{R}$. $E(z)$ extends e^x from $\mathbb{R}$ to $\mathbb{C}$

as analytic fcn.

8) $E(i\theta) = \cos\theta + i\sin\theta$

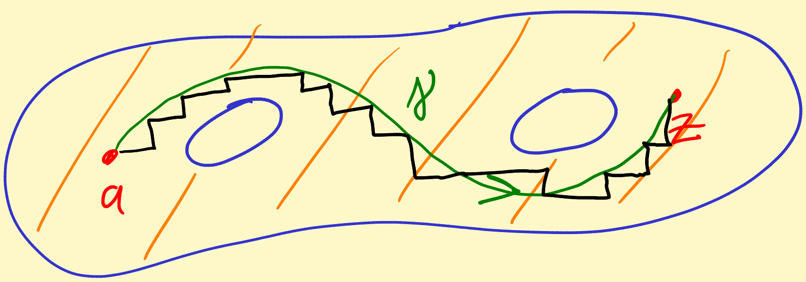
#3 $\Rightarrow$ DeMoivre's formula. #4 $\Rightarrow$ $N<0$.

9) $E(z) = 1 \iff z = n\,2\pi i$, $n \in \mathbb{Z}$.

Only need to prove #3.

Lemma 1 If f analytic on $D_r(a)$ (or $\mathbb{C}$) and $f' \equiv 0$ there, then f is constant there.

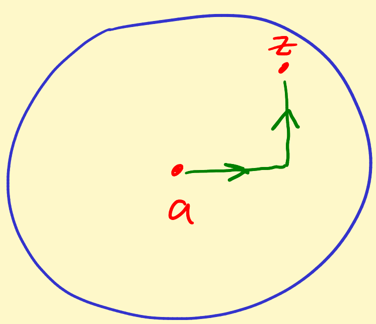
Wish pf $\quad f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \qquad f' \equiv 0 \Rightarrow \begin{cases} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{cases}$



$$\int_\gamma \underbrace{\nabla u}_{\equiv 0} \cdot d\vec{s} = \underbrace{u(z) - u(a)}_{=0}$$

Ouch! Need u C^1-smooth.

Back to pf



Can use freshman calc: $h: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, diff'ble on (a,b), and $h' \equiv 0$ on (a,b).
MVT $\Rightarrow$ h const on $[a,b]$.

See u,v constant on horiz zig and vert zag ✓

Fancier fact: $f' \equiv 0$ on a path-connected open set in $\mathbb{C}$ $\Longrightarrow$ f const.

Lemma 2 If is analytic on $D_R(0)$ (or $\mathbb{C}$) and $f'(z) = K f(z)$ there ($K \in \mathbb{C}$ is a const), then $f(z) = C\,E(Kz)$ where $C = f(0)$.

Pf: $f' - Kf \equiv 0$ Int. factor $E(-Kz)$

Note: $\frac{d}{dz} E(-Kz) = \underbrace{E'(-Kz)}_{E(-Kz)} \underbrace{\frac{d}{dz}[-Kz]}_{-K}$ Chain rule.

$$E(-Kz) \cdot \left[f'(z) - K f(z) \right] \equiv 0$$

$$\underbrace{E(-Kz) f'(z) - K E(-Kz) f(z)}_{\frac{d}{dz}\left[E(-Kz) f(z)\right]} \equiv 0$$

Aha!

Lemma 1 $\Longrightarrow$ $\boxed{E(-Kz) f(z) \equiv c}$, a const.

Plug in $z=0$. See $c = f(0)$. ✓

Stop the proof! Start over using $f(z) = E(Kz)$.

Chain rule shows f satisfies the hyp.

Get red box $\quad E(-kz)E(kz) \equiv c.$

$z=0: \quad c=1.$

Conclude $\quad E(-kz) = 1/E(kz)$ ← cannot vanish

Back to proof: Red box $\Rightarrow$ $f(z) = CE(kz)$ ✓

Pf of #3 $\quad$ Take $f(z) = E(z+w)$.

Think: w = const.

$$f'(z) = E'(z+w)\frac{d}{dz}[z+w]$$
$$= E(z+w)\cdot 1$$
$$= f(z)$$

Lemma 2 $\Rightarrow$ $f(z) = C\cdot E(1\cdot z)$

$C = f(0) = E(0+w) = E(w)$

Done! $\quad E(z+w) = E(w)E(z)$ ✓

Fun: Prove trig identities: $\quad e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$

equate Re and Im parts.

Big fact $\quad f$ analytic on $\Omega \Rightarrow f'$ analytic on Ω

$\Rightarrow f''$ analytic $\Rightarrow \cdots$

Consequence $f = u + iv$ analytic on Ω

$\implies$ u, v are C^∞ smooth there!

Also, u and v are harmonic! $\Delta u \equiv 0$, $\Delta v \equiv 0$

Fun thing to do. $E'(z) = E(z)$

$$\left.\begin{matrix} u_x + i v_x \\ \\ v_y - i u_y \end{matrix}\right\} = u + iv$$

$u = C(y)e^x \rightarrow \boxed{u_x = u}$ $\quad$ $\boxed{v_x = v} \leftarrow v = S(y)e^x$

$v_y = u \qquad -u_y = v$

$S'(y)e^x = C(y)e^x \qquad -C'(y)e^x = S(y)e^x$

$S'(y) = C(y) \qquad -C'(y) = S(y)$

$S''(y) = C'(y) = -S(y) \qquad -C''(y) = S'(y) = C(y)$

See that $S(y)$ and $C(y)$ satisfy $X'' + X = 0$.

Use initial conditions to get that

$$C(y) = \cos y$$

$$S(y) = \sin y.$$