Lecture 5 Complex integration

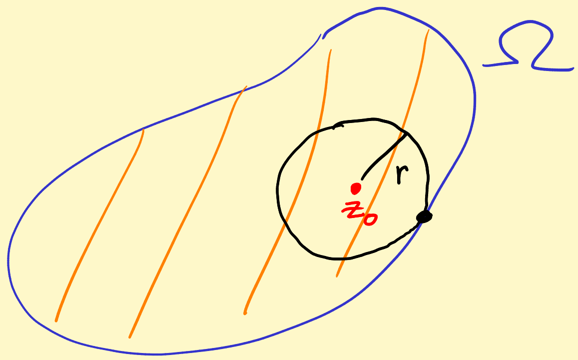
Big facts 1) f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow$ ...

$\Rightarrow$ f' continuous!

2) $f(z) = \sum_0^\infty a_n (z-z_0)^n$ is analytic inside $D_R(z_0)$, ($R = \text{RoC}$). Can diff power series term by term.

$\text{RoC}_{f'} = \text{RoC}_f$.

3) f analytic on open set Ω:

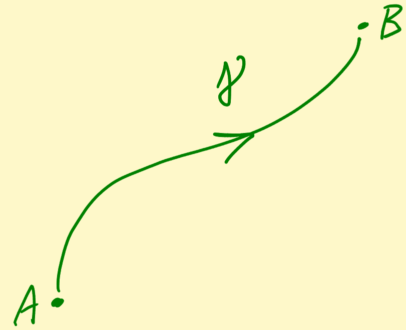


$z_0 \in \Omega$

$$f = \sum_{n=0}^\infty a_n (z-z_0)^n \quad \text{with} \quad \text{RoC} = R \geq r.$$

$$\text{And} \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Integration along paths



$\gamma: z(t) = x(t) + iy(t)$

$a \leq t \leq b$

$z: [a,b] \to \mathbb{C}$

$\text{tr}(\gamma) = \{z(t) : a \leq t \leq b\}$ = "trace of γ"

$$z'(t_0) = \lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0} = \lim [DQ_x] + i \lim [DQ_y]$$

$$= x'(t_0) + i y'(t_0)$$

Complex tangent vector: $T = z'(t)$

Unit complex tangent vector: $\hat{T} = \dfrac{z'(t)}{|z'(t)|}$

<u>Chain rule #1</u>: $g : \Omega_1 \to \Omega_2$ analytic [Ω's open sets]

$f : \Omega_2 \to \mathbb{C}$ analytic

$h(z) = f(g(z))$ analytic on Ω_1 and $h'(z) = f'(g(z))g'(z)$.

<u>Pf</u>: Baby Rudin. Let $x = z$, $|\cdot| =$ modulus.

<u>Chain rule #2</u> f analytic on open set $\Omega \subset \mathbb{C}$

containing $tr(\gamma)$, $z(t)$ diff'ble, then

$$\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$$

complex deriv. new kind

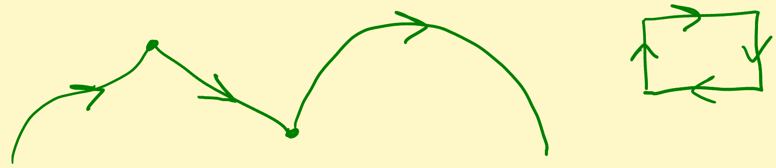
<u>Pf</u>: Baby Rudin again ... or

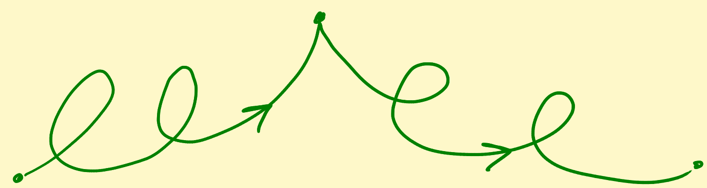
$$f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$$

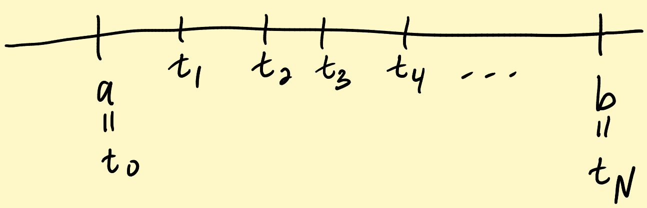
$$\frac{d}{dt} f(z(t)) = [u_x \dot{x} + \overset{-v_x}{\overset{\shortparallel}{u_y}} \dot{y}] + i [v_x \dot{x} + \overset{u_x}{\overset{\shortparallel}{v_y}} \dot{y}]$$

= Complex multiplication! (Red box #1)

Piecewise $\mathcal{C}^1$-curves







$\xrightarrow{z(t)} \mathbb{C}$

$z(t)$ cont on $[a,b]$

$z(t)$ continuously diff'ble on (t_n, t_{n+1}).

$z'(t)$ extends continuously to $[t_n, t_{n+1}]$.

Two kinds of $\int$'s.

Def^n: $$\int_a^b z(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} z(t_n)\,\Delta t$$

$$= \lim [\text{Real part}] + i \lim [\text{Im part}]$$

$$\overset{\text{def}^n}{=} \int_a^b x(t)\,dt + i \int_a^b y(t)\,dt$$

Ex $$\int_0^{2\pi} e^{it}\,dt = \int_0^{2\pi} \cos t\,dt + i \int_0^{2\pi} \sin t\,dt = 0$$

Fund Thm Calc #1 $$\int_a^b z'(t)\,dt = z(b) - z(a)$$

when $z(t)$ is continuously diff'ble on $[a,b]$.

Remark: Standard trick: True for piecewise C' $z(t)$:

$$\int_a^b z'(t)\,dt = \sum_{n=0}^{N-1} \underbrace{\int_{t_n}^{t_{n+1}} z'(t)\,dt}_{z(t_{n+1})-z(t_n)} = z(t_N) - z(t_0)$$

"telescopes"

Pf of Fund Thm Calc #1 Take real and imag parts.

Use freshman calc version of FTC.

Lemma $\left|\int_a^b z(t)\,dt\right| \le \int_a^b |z(t)|\,dt$

Think $\left|\sum z(t_n)\Delta t\right| \le \sum |z(t_n)|\,\Delta t$

Pf $\int_a^b z(t)\,dt = re^{i\theta}$ $\qquad r = \left|\int_a^b z\,dt\right|$

Let $\lambda = \overline{(e^{i\theta})} = e^{-i\theta}$

$$\left|\int_a^b z(t)\,dt\right| = r = \lambda \underbrace{\int_a^b z(t)\,dt}_{re^{i\theta}}$$

$$= \int_a^b \lambda z(t)\,dt$$

check

$$= \underbrace{\int_a^b \mathrm{Re}\left[\lambda z(t)\right] dt}_{r} + i \underbrace{\int_a^b \mathrm{Im}\left[\lambda z(t)\right] dt}_{0}$$

$$= r + i \cdot 0$$

Aha! $r = \left| \int_a^b z(t)\, dt \right| = \int_a^b \underbrace{\mathrm{Re}\left[\lambda z(t)\right]}_{\le |\lambda z(t)| = \underbrace{|\lambda|}_{1} |z(t)|} dt$

$$\le \int_a^b |z(t)|\, dt \quad \checkmark$$

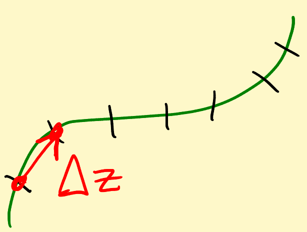
Def^n Suppose $f : tr(\gamma) \to \mathbb{C}$ is continuous

$$\int_\gamma f(z)\, dz \overset{\text{def}^n}{=} \int_a^b f(z(t)) \underbrace{z'(t)\, dt}_{\text{Think } = dz}$$

Konrad Knopf. Dover:

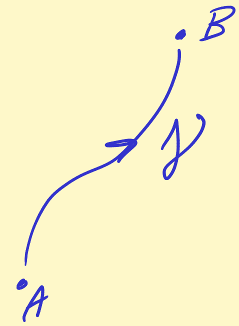
$$\int_\gamma f\, dz = \lim_{\Delta t \to 0} \sum f(z(t_n)) \underbrace{\Delta z_n}_{z(t_{n+1}) - z(t_n)}$$

$$\approx z'(t_n)\, \Delta t_n$$



Fund Thm Calc #2 f analytic on an open set Ω containing $tr(\gamma)$ and f' is cont on Ω.

Then $$\int_\gamma f'(z)\,dz = f(B) - f(A)$$



Pf: $$\int_\gamma f'(z)\,dz = \int_a^b \underbrace{f'(z(t))\,z'(t)}_{\text{Chain rule \#2}\ = \frac{d}{dt} f(z(t))}\,dt$$

$$\underset{\text{FTC \#1}}{=} f(\underbrace{z(b)}_{B}) - f(\underbrace{z(a)}_{A}) \quad \checkmark$$

Basic estimate #2 $$\left|\int_\gamma f\,dz\right| \le \left(\max_{tr(\gamma)} |f|\right) \cdot \text{Length}(\gamma)$$