Lecture 6 Goursat's lemma

HWK 2 due Thurs in GS

Basic estimate: $\left| \int_\gamma f \, dz \right| \leq \left(\underset{z \in tr(\gamma)}{\text{Max}} |f(z)| \right) \text{Length}(\gamma)$

Pf $\left| \int_\gamma f \, dz \right| = \left| \int_a^b f(z(t)) \, z'(t) \, dt \right|$ $\qquad M = \underset{a \leq t \leq b}{\text{Max}} |f(z(t))|$

$$\leq \int_a^b \underbrace{|f(z(t))|}_{\leq M} |z'(t)| \, dt \leq M \underbrace{\int_a^b \underbrace{|z'(t)|}_{\sqrt{(x'(t))^2 + (y'(t))^2}} dt}_{= L = \text{length } \gamma \text{ in } \mathbb{R}^2}$$

$\leq M \cdot L$ ✓

Important fact: $\int_\gamma P(z) \, dz = 0$ if γ is a <u>closed</u> curve and $P(z)$ is a complex poly.

Because $z^n = \frac{d}{dz}\left[\frac{1}{n+1} z^{n+1} \right]$, so

$$\int_\gamma z^n \, dz = \int_\gamma \frac{d}{dz}\left[\frac{1}{n+1} z^{n+1} \right] dz = \left[\frac{1}{n+1} z^{n+1} \right]_{\text{START}}^{\text{END}} = 0$$

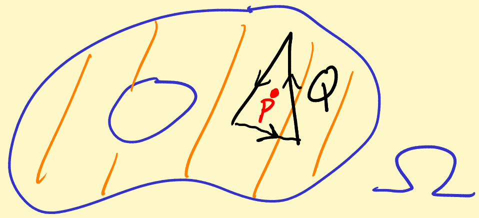
Remark: $n \in \mathbb{Z}$, $n \neq -1$ works same way $[0 \notin tr(\gamma), n < 0]$

Note: $\int_\gamma \underbrace{Az+B}_{\text{Think } \approx f(z) \text{ analytic } \underline{\text{locally}}.} \, dz = 0$, γ closed

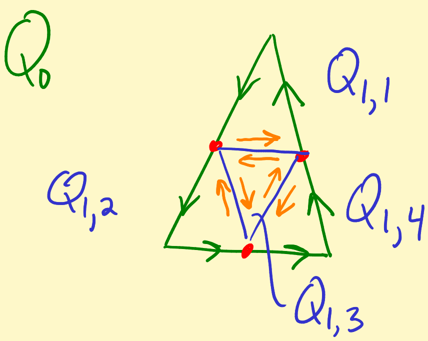
Goursat's lemma $\Omega \subset \mathbb{C}$ open set, $p \in \Omega$, $f : \Omega \to \mathbb{C}$ continuous on Ω and analytic on $\Omega - \{p\}$, Q is a triangle $\Delta \subset \Omega$.

[Notation: Q = triangle = curve going around Q.]

Then $\displaystyle\int_Q f\,dz = 0$.



Pf Case p outside Δ. Write $Q_0 = Q$



Aha! $\displaystyle\underbrace{\int_{Q_0} f\,dz}_{I} = \sum_{j=1}^{4} \int_{Q_{1,j}} f\,dz$

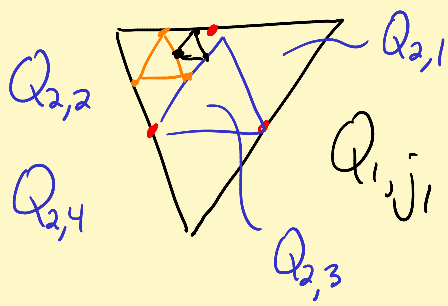
Baby lemma If $I = a_1 + a_2 + a_3 + a_4$, then $\exists\, j$ with $|a_j| \geq \frac{|I|}{4}$.

Pf: $I=0$ case. ✓ $I \neq 0$. $|I| \leq \sum_1^4 |a_j|$

$< \sum_1^4 \frac{|I|}{4} = |I|$ if all $< \frac{|I|}{4}$. ↯

So $\exists\ j_1$ such that $\left|\int_{Q_{1,j_1}} f\,dz\right| \geq \frac{|I|}{4}$.

Repeat!



Get Q_{2,j_2}

$$\left|\int_{Q_{2,j_2}} f\,dz\right| \geq \frac{1}{4}\left|\int_{Q_{1,j_1}} f\,dz\right|$$

$$\geq \frac{1}{4}\cdot\frac{|I|}{4}$$

$$\boxed{Q_{n,j_n} = Q_n}$$

Get sequence $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$

$$\left|\int_{Q_n} f\,dz\right| \geq \frac{|I|}{4^n}$$

Topology lemma $K_n \neq \emptyset$ compact in $\mathbb{R}^2$

$$K_0 \supset K_1 \supset K_2 \supset \cdots$$

Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$. Furthermore, if

$\operatorname{diam}(K_n) \to 0$ as $n \to \infty$, then

$\bigcap_0^{\infty} K_n = \{z_0\}$, a single pt in K_0.

Δ pf easy. Pick corner of each Δ: $w_n \in Q_n$.

Easy to see $\{w_n\}$ is Cauchy. $|w_n - w_m| < \operatorname{diam}(Q_n)$ when $m > n$.

So $w_n \longrightarrow$ some z_0. Q_0 closed. So $z_0 \in Q_0$.

Back to Goursat: $\bigcap_0^\infty Q_n = \{z_0\}$.

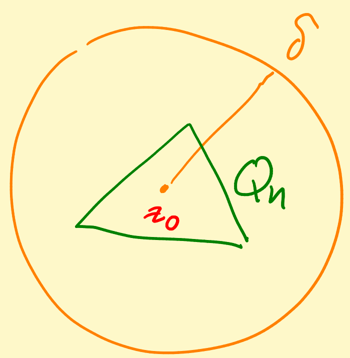
Near $z_0 \neq p$: $\dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + E(z)$

$E(z) \to 0$ as $z \to z_0$.

Define $E(z_0) = 0$.

Continuous on Ω $\longrightarrow$ $f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{B + Az} + E(z)(z - z_0)$

Let $\varepsilon > 0$. $\exists\, \delta > 0$ such that $|E(z)| < \varepsilon$ when $|z - z_0| < \delta$. $\exists\, N$ such that $Q_n \subset D_\delta(z_0)$ when $n > N$.



Now $\displaystyle\int_{Q_n} f\, dz = \underbrace{\int_{Q_n} B + Az\, dz}_{=0} + \int_{Q_n} E(z)(z - z_0)\, dz$ ↑small!

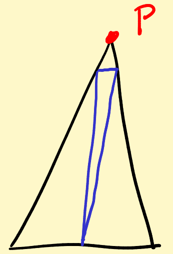
$$\left|\int_{Q_n} f\,dz\right| \le \underbrace{\left(\max_{z\in Q_n}|E(z)|\right)}_{<\,\varepsilon}\ \underbrace{\left(\max_{z\in Q_n}|z-z_0|\right)}_{\le\, \underbrace{\operatorname{diam}(Q_n)}_{\frac{1}{2^n}\operatorname{diam}(Q_0)}}\ \underbrace{\operatorname{Length}(Q_n)}_{\frac{1}{2^n}\operatorname{Length}(Q_0)}$$

$$\le \frac{1}{4^n}\,\varepsilon\ \underbrace{\operatorname{diam}(Q_0)\operatorname{Length}(Q_0)}_{c>0} \leftarrow \text{too small}$$

because $\dfrac{|I|}{4^n} \le \left|\int_{Q_n} f\,dz\right| \le \varepsilon\,\dfrac{1}{4^n}\cdot c$

$|I| \le c\,\varepsilon$, $\varepsilon>0$ arbitrary! So $I=0$. ✓

Case p = vertex



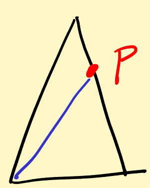
$$\int_Q = \text{Sum } \circlearrowleft \text{ counterclockwise}$$

$$= \int_{\text{Tiny top }\Delta} f\,dz$$

f cont at p, so $|f|\le M$ near p.

$$\left|\int_{\text{tiny}\,\Delta} f\,dz\right| \le M\cdot \operatorname{Length}(\text{tiny}) < \text{arb small } \varepsilon>0.$$

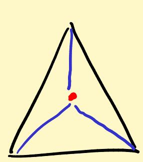
Case $p \in$ edge:



$\int_Q =$ sum of two

reduced to p = vertex case ✓

Case p inside Δ:



reduced to p = vertex case. ✓

Why care about p? Because

$$\underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{\text{analytic } z \neq z_0 \text{ if } f \text{ analytic.}} \xrightarrow[z \to z_0]{} f'(z_0) \quad \text{cont there.}$$

Important in pf of the "Cauchy integral formula"