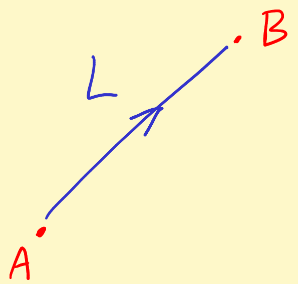
Lecture 7 Cauchy theorem on a convex open set

EX



$$L: \; z(t) = A + t(B-A), \quad 0 \le t \le 1$$
$$z'(t) = (B-A)$$

Important facts: $\int_\gamma f\,dz$ does not depend on the parametrization.

$\int_{-\gamma} f\,dz = -\int_\gamma f\,dz$ where $\gamma:\; z(t), \; a \le t \le b$

$-\gamma:\; z(b + t(a-b)), \; 0 \le t \le 1.$

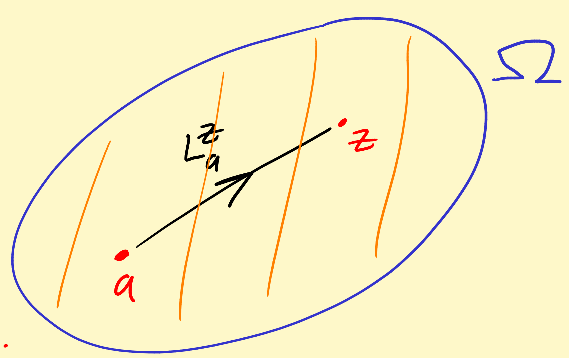
Note: Assume $z'(t) \ne 0$ where curve is C^1-smooth.

Thm Suppose Ω is a convex open set in $\mathbb{C}$,

$f: \Omega \to \mathbb{C}$ is continuous on Ω, $p \in \Omega$,

f is analytic on $\Omega - \{p\}$.

Then $\int_\gamma f\,dz = 0$ for any closed curve γ in Ω.



Fix $a \in \Omega$.

Pf: Plan: cook up a complex antiderivative of f on Ω, i.e., F such that $F'(z) = f(z)$. ← continuous!

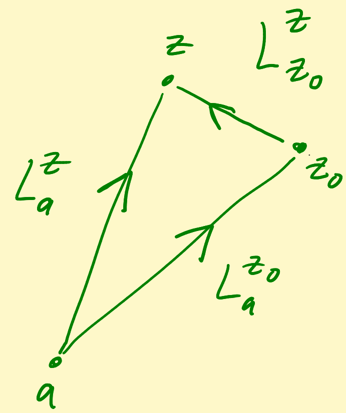
If we had F, $F' = f$ cont $\Rightarrow$ can use Fund Thm Calc #2.

$$\int_{L_a^z} f\,dw = \int_{L_a^z} F'\,dw = F(z) - F(a)$$

Aha! $\frac{d}{dz}\left(\int_{L_a^z} f\,dw\right) = F'(z) = f(z)$. Key: Get F.

Big idea: <u>Define</u> $F(z) = \int_{L_a^z} f(w)\,dw$.

Show $F' = f$. Key: Goursat.



Goursat $$\left(\int_{L_a^{z_0}} + \int_{L_{z_0}^{z}} + \underbrace{\int_{-L_a^{z}}}_{= -\int_{L_a^z} f\,dz}\right) f\,dw = 0$$

Aha! $$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\int_{L_{z_0}^{z}} f\,dw$$

(expect $\to f(z_0)$.)

<u>Baby fact</u> $\int_{L_{z_0}^{z}} c\,dw = c(z - z_0)$.

$$\int_{L_{z_0}^{z}} \frac{d}{dw}[cw]\,dw = [cw]_{z_0}^{z} \checkmark$$

So $(c = f(z_0))$ $$f(z_0) = \frac{1}{z - z_0}\int_{L_{z_0}^{z}} f(z_0)\,dw$$

$$\left| DQ - f(z_0) \right| = \left| \frac{1}{z-z_0} \int_{L_{z_0}^{z}} [f(w) - f(z_0)]\, dw \right|$$

$$\leq \frac{1}{|z-z_0|} \underbrace{\left(\max_{w \in L_{z_0}^{z}} |f(w) - f(z_0)| \right)}_{\text{Continuity : } < \varepsilon \text{ if } |z-z_0| < \delta} \underbrace{\text{Length}}_{|z-z_0|} (L_{z_0}^{z})$$

Cor : Analytic fcns on a convex open set have analytic antiderivatives given by $\int_{L_a^z} f(w)\, dw$, a fixed in Ω.

Back to proof : $F'(z) = f(z)$ ← continuous

$$\int_\gamma f\, dz = \int_\gamma F'(z)\, dz = F(\text{END}) - F(\text{START}) = 0$$

$=$ γ closed

Cauchy integral formula on a convex open set : $z_0 \in \Omega$

f analytic on Ω

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

← g analytic on $\Omega - \{z_0\}$.

← g cont at z_0, cont on Ω

$p = z_0$

Cauchy on convex : $\int_\gamma g(z)\, dz = 0$. γ closed

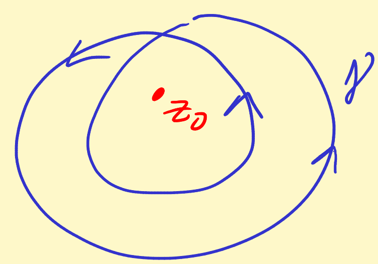
Case $z_0 \notin \operatorname{tr}(\gamma)$: $\displaystyle\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\int_\gamma \frac{1}{z-z_0}\,dz$$

Divide by $2\pi i$

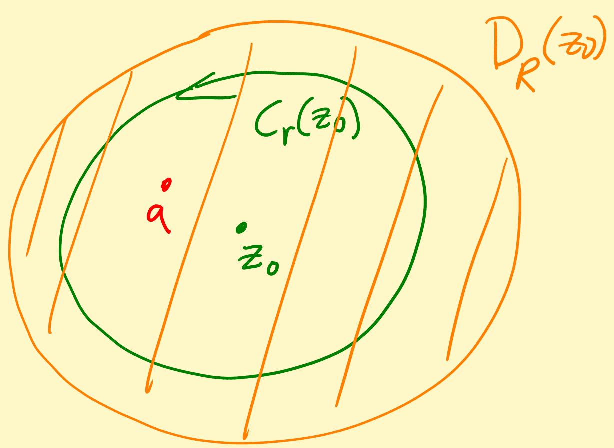
$$f(z_0)\underbrace{\operatorname{Ind}_\gamma(z_0)} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$$

Index or winding # $= \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{1}{z-z_0}\,dz$



$\operatorname{Ind}_\gamma(z_0) = 2$

Cauchy integral formula for a disc



f analytic on $D_R(z_0)$,

$0 < r < R$

$C_r(z_0)$: $z(t) = z_0 + re^{it}$

$0 \le t \le 2\pi$

counterclockwise.

$a \in D_r(z_0)$.

Then $\quad f(a) = \dfrac{1}{2\pi i} \displaystyle\int_{C_r(z_0)} \dfrac{f(z)}{z-a}\, dz$

Lemma $\quad \displaystyle\int_{C_r(z_0)} \frac{1}{z-a}\, dz = \begin{cases} 2\pi i & a \in D_r(z_0) \\ 0 & |a-z_0| > r \end{cases}$

↑ by Cauchy on convex

Pf: $\quad H(a) = \displaystyle\int_{C_r(z_0)} \frac{1}{z-a}\, dz$

Step 1 $\quad H(z_0) = \displaystyle\int_0^{2\pi} \frac{1}{(z_0 + re^{it}) - z_0} \underbrace{\left[ire^{it}\right]}_{z'(t)} dt$

$$= \int_0^{2\pi} i \, dt = 2\pi i$$

Step 2 $\quad H'(a) = \dfrac{d}{da}\left(\displaystyle\int_{C_r(z_0)} \frac{1}{z-a}\, dz \right)$ ← HWK 2 Prob 1 (Can diff under $\int$)

$$= \int_{C_r(z_0)} \frac{1}{(z-a)^2}\, dz$$

Aha! $= \displaystyle\int_{C_r(z_0)} \frac{d}{dz}\left[\frac{-1}{z-a}\right] dz = \left[\frac{-1}{z-a}\right]_{\text{START}}^{\text{END}} = 0$

So $H' \equiv 0$ and $H \equiv$ const. $H(z_0) = 2\pi i$.

So $H \equiv 2\pi i$ on $D_r(z_0)$.

Higher order Cauchy formulas

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

This is how you see that f analytic $\Rightarrow$ f' analytic!

$\Rightarrow$ $f = u + iv$, u, v harmonic, C^∞-smooth.