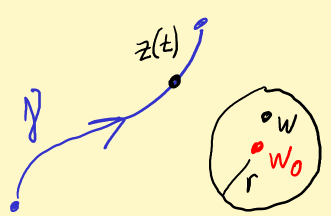
Lecture 8 Liouville's theorem and the Fundamental Theorem of Algebra

Last time: Needed $\frac{d}{dw}\Big[\underbrace{\int_\gamma \frac{1}{z-w}\,dz}_{H(w)}\Big] = \int_\gamma \frac{\partial}{\partial w}\left[\frac{1}{z-w}\right]dz = \int_\gamma \frac{1}{(z-w)^2}\,dz$

Pf
$$\frac{H(w)-H(w_0)}{w-w_0} = \int_\gamma \underbrace{\left(\frac{1}{z-w} - \frac{1}{z-w_0}\right)}_{\frac{w-w_0}{(z-w)(z-w_0)}} dz = \frac{(w-w_0)}{w-w_0}\int_\gamma \frac{1}{(z-w)(z-w_0)}\,dz$$

$$DQ - \underbrace{\int_\gamma \frac{1}{(z-w_0)^2}\,dz}_{I_0} = \int_\gamma \underbrace{\left[\frac{1}{(z-w)(z-w_0)} - \frac{1}{(z-w_0)^2}\right]}_{\frac{w-w_0}{(z-w)(z-w_0)^2}} dz$$

$$\left| DQ - I_0 \right| = |w-w_0| \left| \int_\gamma \frac{1}{(z-w)(z-w_0)^2}\,dz \right|$$



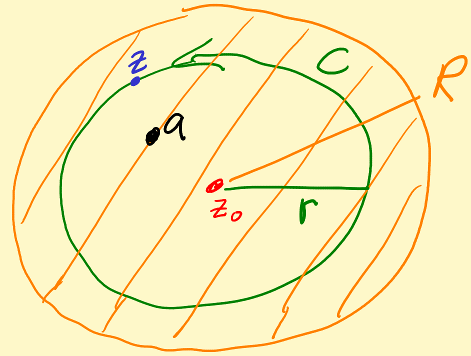
Need $r < \underbrace{\mathrm{dist}(w_0, \mathrm{tr}(\gamma))}_{= \min_{a\le t\le b} |z(t)-w_0| > 0} = d$

Take $r = \frac{d}{2}$. Then $\left| \int_\gamma \frac{1}{(z-w)(z-w_0)^2}\,dz \right| \le \frac{1}{(\frac{d}{2})(d)^2}\,\mathrm{Length}(\gamma)$

See $DQ \to I_0$ as $w \to w_0$.

Can do same routine for $\frac{d}{dw}\int_\gamma \frac{1}{(z-w)^n}\,dz = \int_\gamma \frac{n}{(z-w)^{n+1}}\,dz$

Higher order Cauchy integral formulas on disc



$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Let $a = z_0$. $|f(z)| \le M$ on C.

Basic estimate

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\,\frac{M}{r^{n+1}}\cdot(2\pi r)$$

Cauchy estimates $\quad |f^{(n)}(z_0)| \le \dfrac{n!\,M}{r^n}$

Case $n=1$: $\quad |f'(z_0)| \le \dfrac{M}{r}$, $\quad M = \max_{tr(C)} |f|$

Liouville's thm: A bounded entire fcn must be const.

$|f| \le M$ on $\mathbb{C}$ $\qquad$ f analytic on $\mathbb{C}$

Pf: Pick $z_0 \in \mathbb{C}$. Cauchy est $n=1$: $|f'(z_0)| \le \dfrac{M}{r}$

Aha! Can let $r \nearrow \infty$. Conclude $f'(z_0) = 0$.

z_0 arbitrary. So $f' \equiv 0$. $\Longrightarrow$ $f = $ const on $\mathbb{C}$.

Way false $\mathbb{R} \to \mathbb{R}$. $\sin x$ is bounded. Not const.

Hmmm. $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$ can't be bounded on $\mathbb{C}$.

Basic polynomial estimate $\quad P(z) = a_N z^N + \cdots + a_1 z + a_0$

$N \geq 1$, $\deg = N$ means $a_N \neq 0$. There exists a radius $R > 0$ and real constants $0 < A < B$ such

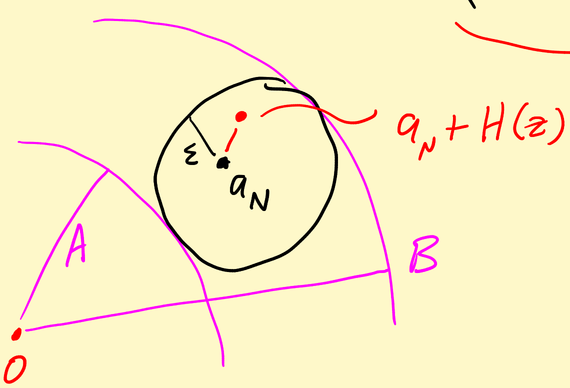
$$A|z|^N \leq |P(z)| \leq B|z|^N$$

when $|z| > R$.

Remark $\quad A < |a_N| < B \leftarrow$ A, B can be as close to $|a_N|$ as desired.

Pf:
$$\frac{P(z)}{z^N} = a_N + \underbrace{\left(\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \right)}_{}$$

$H(z)$, $H(z) \to 0$ as $|z| \to \infty$.



$B = |a_N| + \varepsilon$

$A = |a_N| - \varepsilon$

Books: $\varepsilon = \dfrac{|a_N|}{2}$

$\exists\ R > 0$ such that $|H(z)| < \varepsilon$ when $|z| > R$. ✓

Fund Thm Alg $\quad$ A complex poly $P(z)$ of deg $N \geq 1$ has a root in $\mathbb{C}$.

Pf: Suppose $P(z)$ does not have a zero on $\mathbb{C}$.

Basic est: $A|z|^N \le |P(z)| \le B|z|^N$ when $|z|>R$.

Aha! $\frac{1}{P(z)}$ is entire! And $\left|\frac{1}{P(z)}\right| \le \frac{1}{A|z|^N}$, $|z|>R$.

$\frac{1}{P(z)}$ is a bnd entire fcn: $M = \max_{|z|\le R} \left|\frac{1}{P(z)}\right|$ ← cont on $|z|\le R$.

$|z|\le R$ ↑ closed, bounded set $\subset \mathbb{C}$ compact ✓

Can assume $R>1$. $\left|\frac{1}{P(z)}\right| \le \frac{1}{A|z|^N} \le \frac{1}{AR^N}$ when $|z|>R$.

So $\left|\frac{1}{P(z)}\right|$ is bndd by $\max\left(M, \frac{1}{AR^N}\right)$ on $\mathbb{C}$.

Liouville's $\Rightarrow$ $\frac{1}{P(z)} \equiv$ const. $\Rightarrow$ $P(z) =$ const. $\lightning$

[poly deg $N\ge 1$ not const because $\left.\frac{d^N}{dz^N}P(z)\right|_{z=0} = N!\,a_N \ne 0$.]

Morera's thm Suppose $f:\Omega\to\mathbb{C}$ is continuous on an open set Ω. If $\int_\gamma f\,dz = 0$ for every closed γ in Ω, then f is analytic on Ω.

Note Enough to assume $\int_Q f\,dz$ for Δ's $Q \subset \Omega$

$\bar{\Delta} \subset \Omega$

Pf: Have conclusion of Goursat's on disc $D_r(z_0) \subset \Omega$. (convex)

Define $F(z) = \int_{L_{z_0}^{z}} f\,dw$ on $D_r(z_0)$.

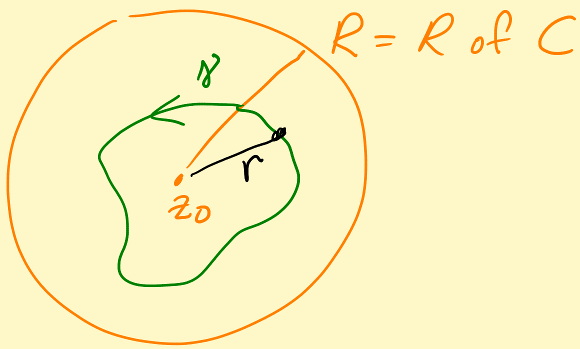
Pf that $F' = f$ on $D_r(z_0)$ only required $\int_{\Delta's} f\,dw = 0$.

So $F' = f$. Today we know F analytic $\Rightarrow$ F' analytic.

So f is analytic on $D_r(z_0)$! So f analytic on Ω.

Cor Power series converge to analytic fcns inside their R of C.

Pf



$\gamma: z(t),\ a \le t \le b$

$|z(t)|$ cont on $[a,b]$, so it has a max $r < R$.

$$f(z) = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} a_n (z-z_0)^n}_{P_N(z)}$$

$P_N(z) \longrightarrow f(z)$ uniformly on $\{z : |z| \le r\}$.

Note: We know $\int_\gamma P_N(z)\,dz = 0$ because $P_N = \frac{d}{dz}(\text{poly})$.

$$\text{So } \left|\int_\gamma f\,dz\right| = \left| \int_\gamma f\,dz - \underbrace{\int_\gamma P_N\,dz}_{0} \right|$$

$$= \left| \int_\gamma (f - P_N)\,dz \right| \le \underbrace{\left(\max_{|z|\le r} |f - P_N|\right)}_{\text{make } < \frac{\varepsilon}{\text{Length}} \text{ for } N \ge N_0} \text{Length}(\gamma)$$

<u>Note</u> Fact: Uniform limits of cont fcns are cont.

Hyp of Morera's ✓. So f is analytic on $D_R(z_0)$.

<u>Defn</u> Seq of cont fcns f_n on Ω <u>converges uniformly on compact subsets</u> to f if, given $\varepsilon > 0$ and compact $K \subset\subset \Omega$, there is an N such that

$$|f_n(z) - f(z)| < \varepsilon \text{ for all } z \in K \text{ when } n \geq N.$$

Write $f_n \twoheadrightarrow f$. <u>Thm</u>: f_n analytic $\twoheadrightarrow f$ then f analytic.