Lecture 9 Analytic fcns are given by convergent power series locally

Know f analytic $\Rightarrow f'$ analytic $\Rightarrow f''$ analytic $\Rightarrow \cdots$

f' continuous $\Leftarrow$ (f'' analytic)

$$f' = \begin{cases} u_x + i v_x & \#1 \\ v_y - i u_y & \#2 \end{cases}$$

$\Rightarrow$ first partials u, v are continuous.

Repeat: See all partials of u, v exist and are continuous, i.e., u, v C^∞-smooth.

Claim: u, v are harmonic: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$\text{C-R eqns} \quad \begin{cases} u_x = v_y & (A) \\ v_x = -u_y & (B) \end{cases}$$

$$\frac{\partial}{\partial x}(A): \quad u_{xx} = v_{xy}$$

$$\frac{\partial}{\partial y}(B): \quad v_{yx} = -u_{yy}$$

v is C^2-smooth!

$u_{xx} = -u_{yy}$ ✓

Taking $\frac{\partial}{\partial y}(A)$ and $\frac{\partial}{\partial x}(B)$ shows v harmonic

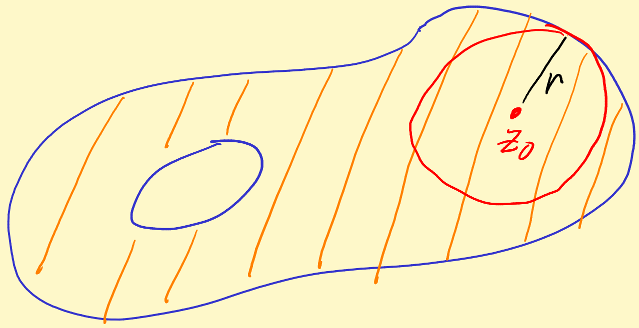
Notation $$\frac{1}{1-z} = \underbrace{1 + z + \cdots + z^N}_{S_N(z)} + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

Know: If $|z| < \rho < 1$, then $|E_N(z)| < \frac{\rho^{N+1}}{1-\rho}$.

Thm f analytic on $D_R(z_0)$, then

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n=\frac{f^{(n)}(z_0)}{n!},$$

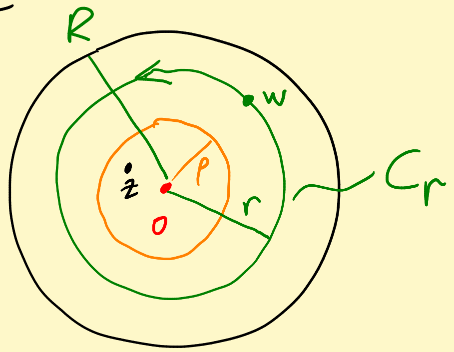
RofC $\geq R$.



f analytic

RoC $\geq r$

Pf Can assume $z_0=0$. Let $0<\rho<r<R$



$$f(z)=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw$$

$$f(z)=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}\underbrace{\frac{1}{1-\frac{z}{w}}}_{S_N\left(\frac{z}{w}\right)+E_N\left(\frac{z}{w}\right)}dw$$

$$=\underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}\left[1+\left(\frac{z}{w}\right)+\cdots+\left(\frac{z}{w}\right)^N\right]dw}_{\sum_{n=0}^{N}\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{n+1}}\,dw\right)z^n}+\underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}E_N\left(\frac{z}{w}\right)dw}_{\mathcal{E}_N(z)}$$

$$\sum_{n=0}^{N}\underbrace{\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{n+1}}\,dw\right)}_{\frac{f^{(n)}(0)}{n!}}z^N$$

Higher order Cauchy!

$$|\mathcal{E}_N(z)|\leq\frac{1}{2\pi}\frac{M}{r}\frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}}(2\pi r)$$

where $M=\max_{C_r}|f|$.

$\mathcal{E}_N(z)\longrightarrow 0$ uniformly

when $|z|<\rho<r<R$.

Can squeeze $\rho, r \nearrow R$ to see series converges on $D_R(z_0=0)$.

So $RoC \geq R$.

Cor f analytic on $D_R(z_0)$ where $R = R$ of C.

Know f' analytic on $D_R(z_0)$. Claim $RoC_{f'} = RoC_f$.

Pf f' analytic on $D_{R_f}(z_0) \Rightarrow RoC_{f'} \geq RoC_f$

Claim $RoC_{f'} \leq RoC_f$ too, so they are $=$.

Suppose power series for f' converges on a bigger disc

$D_R(z_0)$, $R > RoC_f$, converges to $\tilde{\varphi}$ analytic on $D_R(z_0)$.

Cook up F analytic antiderivative of $\tilde{\varphi}$ on convex

$F' = \tilde{\varphi}$. Then $\frac{d}{dz}\left[f - F\right] = f' - F' \equiv 0$ on $D_{R_f}(z_0)$.

So $f - F = c$ on $D_{R_f}(z_0)$

$$\underbrace{f}_{R\,of\,C < R} = \underbrace{F + c}_{R\,of\,C \geq R}$$

Power series are uniquely determined by Taylor's formula.
Should have done before last argument.

Pf Suppose $f(z) = \sum_0^\infty a_n (z-0)^n$ on $D_R(0)$ converges unif on

$D_r(0)$, $0 < r < R$. $0 < \rho < r < R$

$$f^{(N)}(0) = \frac{N!}{2\pi i}\int_{C_\rho} \frac{f(z)}{z^{N+1}}\,dz = \frac{N!}{2\pi i}\int_{C_\rho} \frac{1}{z^{N+1}}\left[\sum_{n=0}^{\infty} a_n z^n\right] dz$$

$$= \frac{N!}{2\pi i}\sum_{n=0}^{\infty} a_n \underbrace{\int_{C_\rho} z^{n-N-1}\,dz}_{=0} = \frac{N!}{2\pi i}\, a_N\,(2\pi i) \checkmark$$

unif conv $\Rightarrow$ can exchange $\int, \Sigma$

except when $n-N-1=-1$
$n=N$

Can differentiate power series term by term:

$$f' = \sum_{n=0}^{\infty} b_n (z-z_0)^n \quad \text{where} \quad b_n = \frac{(f')^{(n)}(z_0)}{n!}$$

$$= \frac{f^{(n+1)}(z_0)}{(n+1)\,n!}\cdot(n+1) = \left[\frac{f^{(n+1)}(z_0)}{(n+1)!}\right](n+1)$$

$$b_n = (n+1)\,a_{n+1} \checkmark$$

Next time: Zeroes of analytic fcns behave like zeroes of polynomials!

$$f(z) = a_0 + a_1(z-z_0) + \cdots + a_{N-1}(z-z_0)^{N-1} + a_N(z-z_0)^N + \cdots$$

$a_0 = f(0)$, $a_1 = \frac{f'(0)}{1!}$, $a_{N-1} = \frac{f^{(N-1)}(0)}{(N-1)!}$

Suppose $f(0) = 0$, $\frac{f'(0)}{1!} = 0$, $\frac{f^{(N-1)}(0)}{(N-1)!} = 0$

a_N: not zero

$$= a_N(z-z_0)^N + \cdots$$

$$= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + a_{N+2}(z-z_0)^2 + \cdots\right]}_{F(z)}$$

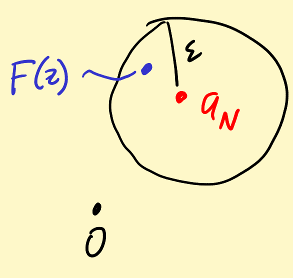
Hmmm. Power series for $F(z)$ converges for <u>exactly</u> the same z as series for $f(z)$. $R_F = R_f$.

$$f(z) = (z-z_0)^N F(z) \qquad R = \text{Rof}_f = \text{Rof}_F.$$

F is analytic on $D_R(z_0)$ <u>and</u> $F(z_0) = a_N \neq 0$.

Consequently, zero of f at z_0 must be <u>isolated</u>:
There is a $\delta > 0$ $(0 < \delta \le R)$ such that z_0 is <u>only</u> zero of f in $D_\delta(z_0)$.

[F = power series is analytic, so <u>continuous</u>.



$\varepsilon < |a_N|$ $\qquad$ $|F(z) - a_N| < \varepsilon$ when $|z-z_0| < \delta$.]

zeroes of analytic fcns can't pile up <u>inside</u> Ω.