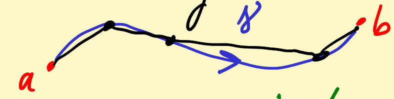
Lecture 10 Zeroes of analytic functions

HWK 3 due Thurs.

Connected open sets are called <u>domains</u>.

<u>Def^n 1</u> An open set Ω is <u>path-connected</u> if any two points in Ω can be joined by a piecewise C^1-curve <u>in</u> Ω.

Topology: Curve γ connecting a to b in Ω merely continuous. Equiv in open sets in $\mathbb{R}^2$:



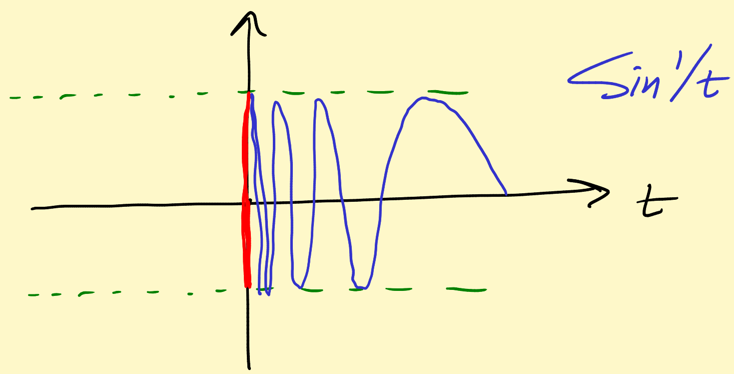
Chords.

<u>Def^n 2</u> Ω open is <u>connected</u> if $\Omega = U_1 \cup U_2$, both open, $U_1 \cap U_2 = \emptyset$,

then one of U_i's $= \Omega$ and other $= \emptyset$.

<u>Fact</u>: Def^n 1 and 2 are <u>equivalent</u> for open sets in $\mathbb{C}$.

Topology ex:



Graph $\cup$ {red line} is connected, but <u>not</u> path-conn.

<u>Lemma</u> If f is analytic on $D_r(z_0)$ and $f(z_0) = 0$, then

either A) $f \equiv 0$ on $D_r(z_0)$, or

B) z_0 is an <u>isolated</u> zero.

<u>Pf</u> $f(z) = a_N (z - z_0)^N + \cdots$

first non-zero Taylor coeff

$$= (z-z_0)^N \underbrace{\left[a_N + a_N(z-z_0) + \cdots \right]}_{F(z), \quad F(z_0) = a_N \neq 0.}$$

Def^n f has a zero of order (or multiplicity) N at z_0.

Thm Identity theorem Suppose f analytic on a domain Ω. Let $Z_f = \{ z \in \Omega : f(z) = 0 \}$. Then either

A) $f \equiv 0$, or

B) Z_f has no limit points in Ω.

Pf: Suppose $z_0 \in \Omega$ is a limit point of Z_f, meaning

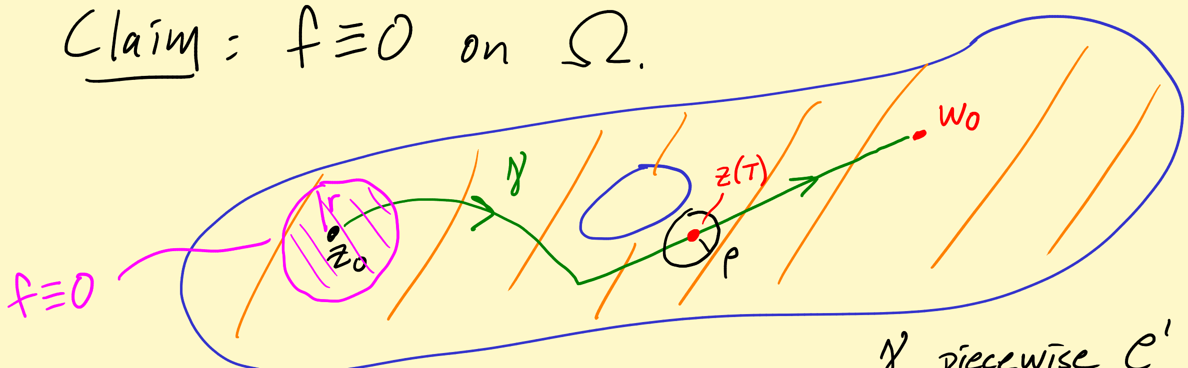
$\exists$ seq $\{z_n\}_{n=1}^{\infty}$ in Z_f with $z_n \neq z_0$ for all n and

$\lim\limits_{n\to\infty} z_n = z_0$. Note: f analytic $\Rightarrow$ f cont, so

$f(z_0) = \lim\limits_{n\to\infty} f(z_n) = 0$. Ω open. So $\exists\ r > 0$

such $D_r(z_0) \subset \Omega$. Lemma $\Rightarrow$ $f \equiv 0$ on $D_r(z_0)$.

Claim: $f \equiv 0$ on Ω.



Pick $w_0 \in \Omega$. Get curve γ joining z_0 to w_0 in Ω ($tr(\gamma) \subset \Omega$).

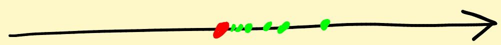
γ piecewise C^1: $z(t)$, $a \le t \le b$.

Let $T = \sup\{\tilde{\tau} : f(z(t)) = 0 \text{ for } a \le t < \tilde{\tau}\}$

$T > a$. <u>Claim</u>: $T = b$. Easy to see $z(T)$ is <u>not</u> an isolated zero of f. As at z_0, get $\rho > 0$ $f \equiv 0$ on $D_\rho(z(T)) \subset \Omega$. T is <u>not</u> the sup if $T < b$. $\lightning$ So $T = b$. ($f(w_0) = 0$ too by continuity.)

Alternate pf: Use $\Omega = U_1 \cup U_2$ open, $U_1 \cap U_2 = \emptyset$ with $U_1 = \{w \in \Omega : w \text{ is a limit pt of } Z_f\}$. $U_2 = \Omega - U_1$. Use Lemma to show U_1 and U_2 are both open.

<u>Consequence</u> e^z is the <u>unique</u> extension of e^x from $\mathbb{R}$ to $\mathbb{C}$.



every pt in $\mathbb{R}$ is a limit pt of $\mathbb{R}$ in $\mathbb{C}$.

<u>Cor</u> If two analytic fcns on a domain Ω agree on a set with a limit point in Ω, then they are the same on Ω.

<u>Consequence</u> Trig identities hold for <u>complex</u> angles!

$$\sin 2\theta = 2\cos\theta \sin\theta$$

holds $\theta \in \mathbb{C}$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}$$

$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta$ $\left\{ \begin{array}{l} \text{first } \alpha = z. \\ \text{then } \beta = w \in \mathbb{C}. \end{array} \right.$

Identity thm way false $\mathbb{R} \to \mathbb{R}$:

$$h(t) = \begin{cases} 0 & , t = 0 \\ e^{-1/t^2} \sin 1/t^2 & , t \neq 0 \end{cases}$$

is C^∞-smooth on $\mathbb{R}$. Zeroes piles up at $t=0$, but $h \not\equiv 0$. Classic example : $h \neq$ Taylor series at $t_0 = 0$.

<u>Remark</u> Can "factor out" finitely many zeroes :

$$f(z) = (z-z_1)^{m_1}(z-z_2)^{m_2} \cdots (z-z_N)^{m_N} G(z)$$

with G analytic on Ω, $\neq 0$ at z_n's.

<u>EX</u> $\sin \frac{1}{z}$ has zeroes at : $\frac{1}{z} = n\pi$, $n \in \mathbb{Z}$.

$$z = \frac{1}{n\pi}$$

Hmmm. zeroes piling up at $z=0$. ???

$\Omega = \mathbb{C} - \{0\}$ open, conn. Ok for zeroes to pile up at boundary pts.