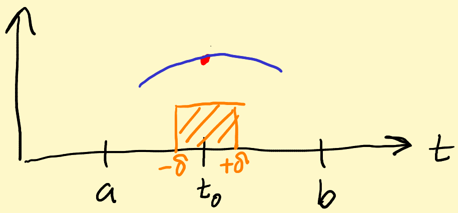
Lecture 11 The Maximum principle

HWK 4 due Thurs 9/26 ← two weeks

Calculus lemmas $h: [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and $h(t) \geq 0$ on $[a,b]$. Then $\int_a^b h(t) = 0 \implies h \equiv 0$ on $[a,b]$.

Pf



Suppose $h(t_0) > 0$, $t_0 \in [a,b]$.

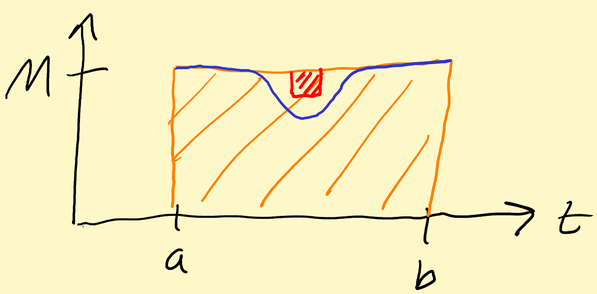
$\varepsilon = \frac{h(t_0)}{2}$, get $\delta > 0$.

$\int > \varepsilon \cdot 2\delta > 0$ ↯

Lemma 2 $h: [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and $h(t) \leq M$ on $[a,b]$.

Then $\int_a^b h(t)\,dt = M(b-a) \implies h \equiv M$ on $[a,b]$.

Pf Use $M-h$ in place of h in Lemma 1.



Area $= M(b-a)$

Key Cauchy $\int$ formula $\implies$ averaging prop $\implies$ max princ.

$C_r(z_0)$: $z(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ $\qquad z'(t) = 0 + ire^{it}$

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(z_0 + re^{it}) - z_0} \cdot ire^{it}\, dt$$

$$= \frac{1}{2\pi r} \int_0^{2\pi} f(z_0 + re^{it})\, \underbrace{r\,dt}_{ds} \leftarrow \text{ave prop}$$

ds ← arc length

$u(z_0) + iv(z_0) = (\text{Ave } u) + i(\text{Ave } v)$

Note Re f, Im f also satisfies ave prop $\underset{\text{soon}}{\Longrightarrow}$ harmonic fcns do too $\Rightarrow$ max princ for harm fcns.

Max princ #1 Suppose f is analytic on a domain Ω. If $|f|$ attains a local max at a pt z_0 in Ω

(meaning, $\exists\ \varepsilon > 0$ with $D_\varepsilon(z_0) \subset \Omega$ and

$|f(z)| \leq |f(z_0)|$ for all $z \in D_\varepsilon(z_0)$), then f is constant on Ω.

Pf Suppose $|f|$ has a local max at z_0, $D_\varepsilon(z_0) \subset \Omega$.

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})\,dt \qquad 0 < \rho < \varepsilon.$$

$$M = |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{it})|\,dt \leq \frac{1}{2\pi}\cdot M \cdot 2\pi = M$$

Aha! See $\int_0^{2\pi} |f(z_0 + \rho e^{it})|\,dt = M \cdot 2\pi$

Calc lemma #2 $\Rightarrow$ $|f(z_0 + \rho e^{it})| \equiv M$, $0 \leq t \leq 2\pi$,

and $0 \leq \rho < \varepsilon$.

Lemma $|f| \equiv M$ on $D_\varepsilon(z_0) \Rightarrow f \equiv$ const there.

Pf: (*) $u^2 + v^2 \equiv M^2$ HWK 3 $\rho(u,v) = u^2 + v^2 - M^2$

Case $M=0$ $\quad f \equiv 0$ $\checkmark$

Case $M>0$

$$\frac{\partial}{\partial x}(*): \quad 2u\,u_x + 2v\,v_x = 0$$

$$\frac{\partial}{\partial y}(*): \quad 2u\,u_y + 2v\,v_y = 0$$

C-R eqns

$$\begin{bmatrix} u_x & v_x \\ \underset{=-v_x}{u_y} & \underset{=u_x}{v_y} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

$\uparrow$ not the zero vector!

So $\det = 0$

$$\underbrace{(u_x)^2 + (v_x)^2}_{|f'|^2} = 0$$

Red box #1: $f' = u_x + i v_x$

So $f' \equiv 0$ on $D_\varepsilon(z_0)$. Identity thm $\Rightarrow f' \equiv 0$ on Ω

$\Rightarrow f$ constant on Ω. $\checkmark$

Max princ #2 Suppose Ω is a bounded domain ($\Omega \subset D_R(0)$ for some $R>0$), and f is continuous on the closure $\overline{\Omega}$ of Ω, and f is analytic on Ω. Then $|f|$ attains its max value on $\overline{\Omega}$ at a pt in the boundary of Ω.

[$\exists\ z_0$ in the bndry of Ω with $|f(z_0)| = \underset{\Omega}{\text{Max}}\, |f|$.

Consequently : $\underset{\overline{\Omega}}{\text{Max}}\, |f| = \underset{\text{bndry}}{\text{Max}}\, |f|$.]

Pf: f cont $\Rightarrow |f|$ cont.

$|f|$ cont on $\overline{\Omega}$, a closed and bounded set in $\mathbb{R}^2$, i.e., a <u>compact</u> set. A cont fcn on a compact set <u>assumes</u> its max value on the set, say at z_0.

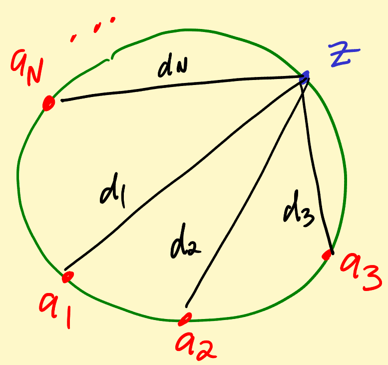
<u>Case $z_0 \in \Omega$</u> Max princ #1 $\Rightarrow f$ const on Ω.

$\Rightarrow f$ const on $\overline{\Omega}$ by continuity. See max is <u>attained</u> at a pt in the bndry.

<u>Case z_0 in bndry</u>. Done!

<u>Remark</u>: z^n shows there is no min princ ($z=0$), <u>but</u> if f is non-vanishing on Ω, there <u>is</u> a min princ. <u>Pf</u> Apply max princ to $1/f$.

<u>Problem</u>



unit circle

Show $\exists\ z$ with $|z|=1$ such that $\prod_{n=1}^{N} d_n = 1$.

Hmmm. $d_n = |z - a_n|$

Aha! $f(z) = \prod_{n=1}^{N} (z - a_n) \leftarrow$ poly deg $N \geq 1$. Not const.

$f(z) = z^N + \cdots \pm \prod_1^N a_n$

$|z| = 1$: $|f(z)| = \prod_{n=1}^{N} d_n$

$$|f(0)| = \left| \prod_1^N (-a_n) \right| = 1$$

Since $f \not\equiv$ const, 1 is not the max modulus.

Max modulus is > 1, and occurs at some z with $|z| = 1$.

Note : $|f(a_n)| = 0$, $n = 1, 2, \ldots, N$.

Say w_0 with $|w_0| = 1$ is the max $|f(w_0)| = M > 1$.

Aha! Intermediate Value Thm $\Rightarrow \exists$ pt z_0 on $|z| = 1$ with $|f(z_0)| = 1$. ✓

Cont fcn $|f(e^{it})|$, $0 \leq t \leq 2\pi$.