Lecture 12 Harmonic fcns and harmonic conjugates

Def^n A harmonic fcn on a domain Ω is a real valued $\mathcal{C}^2$-smooth fcn such that $\Delta u \equiv 0$ on Ω. $\left[\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$

Important tool If u is harmonic, $u_x - i u_y$ is analytic.

Pf $U = u_x$ C-R eqns $U_x \overset{?}{=} V_y$ $u_{xx} \overset{?}{=} -u_{yy}$ ✓ $\Delta u \equiv 0$.

$V = -u_y$ $U_y \overset{?}{=} -V_x$ $u_{yx} = -(-u_{xy})$ ✓

$\mathcal{C}^1$-smooth need u $\mathcal{C}^2$-smooth ✓

Motivation: If $f = u + iv$ analytic, then

$$f' = \begin{cases} u_x + i v_x & \#1 \\ v_y - i u_y & \#2 \end{cases} = u_x - i u_y \longleftarrow !$$

Hmmm. Prob. Given harm u, find v with $f = u + iv$ analytic. If we had v, $f' = u_x - i u_y$. Big idea: Find anti-deriv of $u_x - i u_y$ to get v.

Cor: If u is $\mathcal{C}^2$-smooth harmonic, then u is $\mathcal{C}^\infty$-smooth!

Pf: $u_x - i u_y$ analytic $\Longrightarrow$ u_x, u_y $\mathcal{C}^\infty$-smooth $\Rightarrow$ u, v $\mathcal{C}^\infty$ smooth.

Cor If harmonic u on a domain vanishes on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

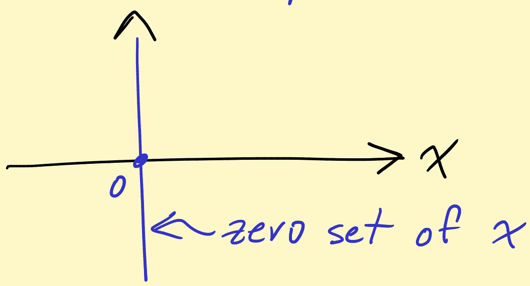
Pf: $u_x - i u_y$ analytic on Ω and $\equiv 0$ on $D_\varepsilon(z_0)$.

$\Longrightarrow$ $u_x - i u_y \equiv 0$ on Ω, i.e., $\nabla u \equiv 0$ on Ω.

$\Rightarrow u$ const on Ω. $u(z_0)=0 \Rightarrow u \equiv 0$.

Can zeroes of harm fcns have limit pts and not be $\equiv 0$?

EX: $u = \operatorname{Re} z = x$



No "Identity thm" for harm fcns.

Future Cauchy thm If f analytic on a simply connected (no holes) domain Ω and γ is a closed curve in Ω, then

$$\int_\gamma f\, dz = 0.$$

Cor If u is harmonic on a domain Ω on which Cauchy thm holds, then u has a harmonic conj on Ω.

Pf: Case: Ω = convex open set. Cauchy thm holds.

Hmmm. $f = u + iv$. $f' = u_x - iu_y$

Aha! Get F such that $F' = \underbrace{u_x - iu_y}_{g}$ via

$$F(z) = \int_{L_a^z} g(w)\, dw.$$

a fixed

Know $\underset{U_x - iU_y}{F'} = g = u_x - iu_y$

Write $F = U + iV$.

So $\nabla \tilde{U} = \nabla u$ on Ω. So $\tilde{U} = u + c$, c const.

<u>Claim</u>: $\tilde{V}$ is a harm conj for u.

$$u + i\tilde{V} = \underbrace{(u+c)}_{\tilde{U}} + i\tilde{V} - c \quad \text{analytic!} \ \checkmark$$

<u>General case</u> on Cauchy thm domain.

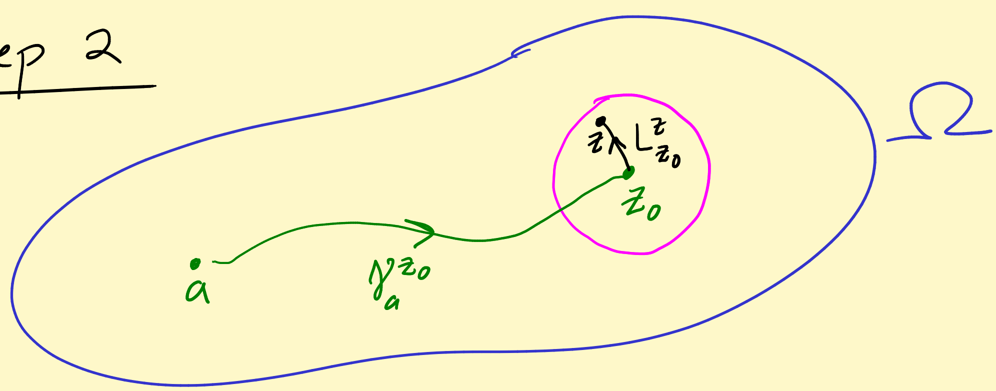
"Define" $\quad F(z) = \int_{\gamma_a^z} g(w)\,dw$

<u>Step 1</u> $\quad F$ is <u>well</u> defined

because $\quad F(z) - \tilde{F}(z) = \underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z} \right)}_{\int_\Gamma \,:\ \Gamma = (\gamma_a^z) \cup (-\tilde{\gamma}_a^z)} g(w)\,dw$

Γ closed. So $\int = 0$.

<u>Step 2</u>



$D_\rho(z_0) \subset \Omega$
open

Aha! $\quad F(z) = \int_{\gamma_a^{z_0} \cup L_{z_0}^z} g\,dw = \underbrace{\int_{\gamma_a^{z_0}}}_{\text{const}} + \underbrace{\int_{L_{z_0}^z}}_{\frac{d}{dz} = g}$

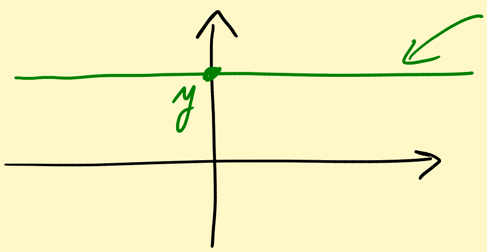
So $F' = g$. Rest same.

Cooking up harm conj on $D_R(z_0)$ or $\mathbb{C}$.

Lemma $\quad \dfrac{\partial u}{\partial x} \equiv 0 \iff u = $ fcn of y alone.

$\qquad \dfrac{\partial u}{\partial y} \equiv 0 \iff \quad$ " $\quad x$

Pf



$\frac{\partial}{\partial x} u(x,y) \equiv 0 \implies u$ const along horiz lines.

So $u(x,y) = C(y)$.

and $\frac{\partial}{\partial x} C(y) \equiv 0$.

Cor: $\quad \dfrac{\partial u_1}{\partial x} \equiv \dfrac{\partial u_2}{\partial x} \iff u_1 = u_2 + (\text{fcn of } y \text{ alone})$

etc

Prob Find a harm conj for $u = x^2 - y^2 + x + y$ on $\mathbb{C}$.

Step 1 Check $\Delta u \equiv 0$. ✓

Step 2 C-R eqns $\quad \left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \quad \begin{aligned} v_x &= -u_y \quad (A) \\ v_y &= u_x \quad (B) \end{aligned}$

Looking for v with $\nabla v = \vec{F} = (-u_y, u_x)$.

MATH 261 $\quad \mathbb{R}^3$: Need $\operatorname{Curl} \vec{F} = 0$ to get v.

$\qquad \mathbb{R}^2$: Need $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$

$$\nabla v = \vec{F}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \qquad \begin{aligned} v_{yx} &= (F_1)_y \\ \| \\ v_{xy} &= (F_2)_x \end{aligned}$$

Check

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(-u_y)$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(u_x)$$

$$-u_{yy} \stackrel{?}{=} u_{xx} \qquad \Delta u = 0$$

Back to prob.

Want $\begin{cases} v_x = 2y-1 & \text{from (A)} \\ v_y = 2x+1 & \text{from (B)} \end{cases}$

<u>First use (A)</u>

$$v = \underbrace{\int 2y-1\,dx}_{\text{treat } y \text{ like a const}} = 2xy - x + C(y)$$

$C(y)$: arb fcn of y.

Why

$$\frac{\partial}{\partial x}\Big[\underbrace{v - (2xy - x)}_{= C(y)}\Big] \equiv 0$$

<u>Next use (B)</u> in a different way.

Got $v = 2xy - x + C(y)$

Want $\dfrac{\partial}{\partial y}\big(2xy - x + C(y)\big) = 2x+1$ (B)

$$2x - 0 + C'(y) \overset{\text{want}}{=} 2x+1$$

Miracle! All x's cancel. [because $\Delta u \equiv 0$]

$$\boxed{C'(y) = 1}$$

So $C(y) = y + k$, k const.

Get $v = 2xy - x + (y + k)$

<u>Fact</u> Harmonic fcns <u>unique</u> up to + const.