Lecture 13 Complex logarithms, isolated singularities

Danger of having holes in Ω for existence of harmonic conjugates:

EX $\quad \operatorname{Log} z = \operatorname{Ln}|z| + i \operatorname{Arg} z$ is analytic on $\mathbb{C} - (-\infty, 0]$. Principal Log fcn

$\operatorname{Ln}|z|$ is harmonic on $\mathbb{C} - \{0\}$. ← hole at $z=0$.

If v is a harmonic conj for $\operatorname{Ln}|z|$ on $\mathbb{C}-\{0\}$, then it is on $\mathbb{C} - (-\infty, 0]$ too, so $\underset{\text{no jump}}{v} = \underbrace{\operatorname{Arg} z + C}_{\text{jump along the "cut" } (-\infty,0]}$ there. ↯

Fact $\quad v, \tilde{v}$ are both harm conj for harm fcn u on a domain Ω, then C-R eqns $\Rightarrow \nabla v \equiv \nabla \tilde{v} \Rightarrow v = \tilde{v} + C$, C const.

Complex log fcns: $\quad w = e^{z} = e^{x+iy} = \underset{r}{e^{x}} \; \underset{e^{i\theta}}{e^{iy}}$

See $\quad e^{x} = |w|$

so $\quad x = \operatorname{Ln}|w|$ ← x is unique

and $\quad y \in \arg w$

$z = \operatorname{Ln}|w| + i\theta, \quad \theta \in \arg w$. ∞ many $\log w$'s!

Def^{n} $\quad \underset{\text{Principal log fcn}}{\operatorname{Log} w} = \operatorname{Ln}|w| + i \underset{\text{Principal arg}}{\operatorname{Arg} w}$ ← $-\pi < \operatorname{Arg} w \le \pi$

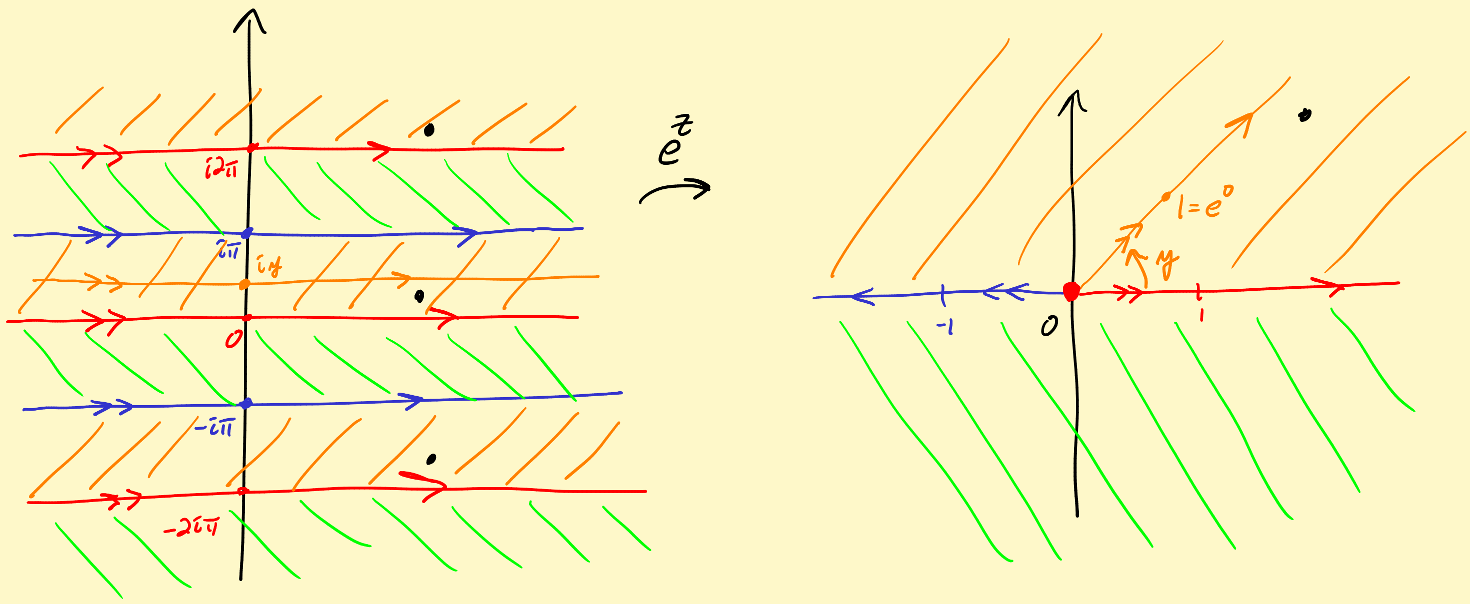
$$\log w = \{ \operatorname{Ln}|w| + i\theta \;:\; \theta \in \arg w \}$$

$$\log_{\alpha} w = \operatorname{Ln}|w| + i\theta \text{ where } \theta \in \arg w \text{ with } \alpha < \theta < \alpha + 2\pi$$

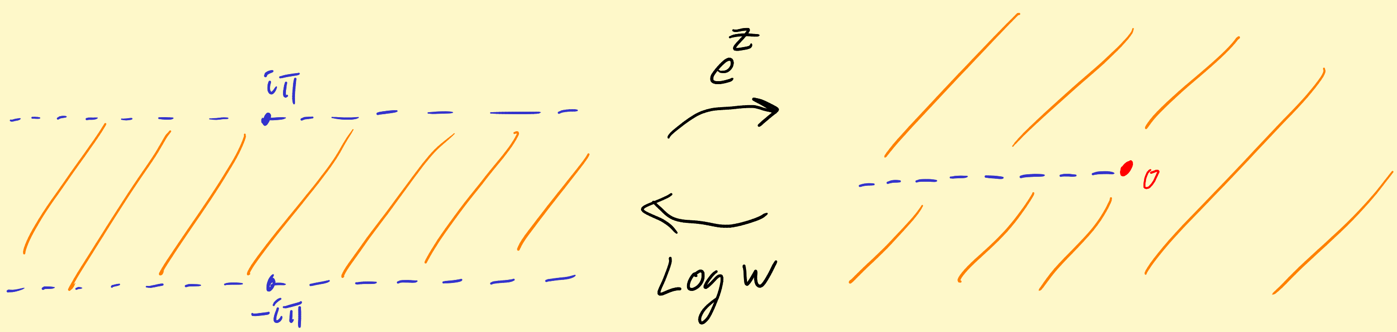
Graph of e^z

$e^{x+iy} = e^x e^{iy}$ $\qquad -\infty < x < \infty$

$0 < e^x < \infty$



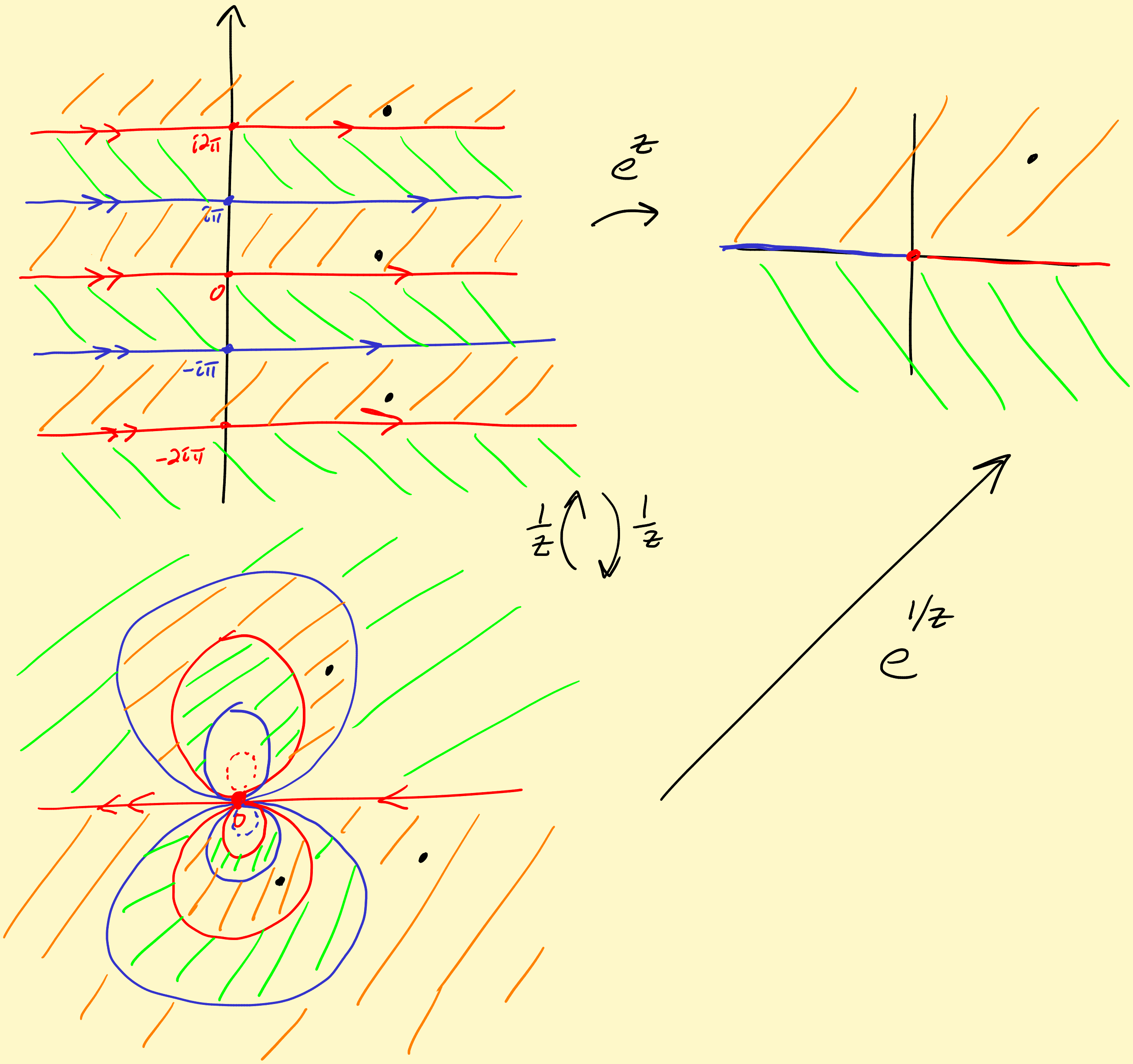
e^z maps $\mathbb{C}$ ∞-to-one to $\mathbb{C}-\{0\}$.



e^z maps $\{z : -\pi < \operatorname{Im} z < \pi\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} - (-\infty, 0]$

e^z has a mind blowing singularity "at ∞", meaning

$e^{1/z}$ has a mind blowing sing at $z=0$.

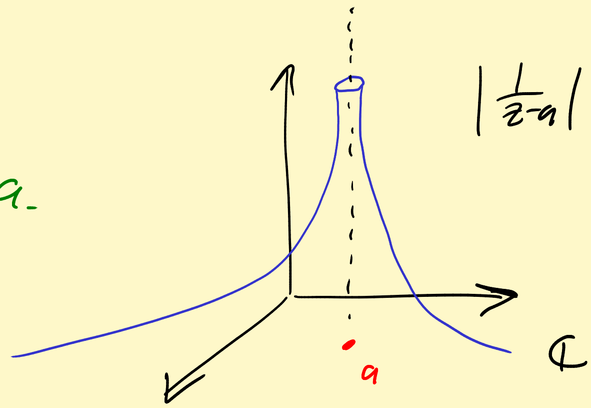


Pick small $\varepsilon > 0$. $e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$ ∞-to-one onto $\mathbb{C} - \{0\}$! 0 is called an "essential singularity."

Other types of behavior at an "isolated singularity" of an analytic fcn.

1) $\frac{1}{z^3}$ "blows up" at $z=0$, meaning that

$\left|\frac{1}{z^3}\right| \longrightarrow \infty$ as $|z| \to 0$. $\frac{1}{z^3}$ has a "pole" at $z=0$.



2) $\frac{1}{z-a}$ has a pole at $z=a$.

3) $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = \frac{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}{z}$$

$$= \underbrace{1 - \frac{z^2}{3!} + \cdots}_{\text{entire!}} \longleftarrow$$ Same RoC as Taylor series for $\sin z$, i.e., $R=\infty$

$z=0$ is a "removable sing":
$$h(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z=0 \end{cases}$$

is analytic near 0.

Fact These are the only 3 possibillities.

Def^n f has an isolated sing at z_0 means $\exists\, \rho > 0$ such that f is analytic on $D_\rho(z_0) - \{z_0\}$.

Def^n z_0 is a pole if $\lim_{z \to z_0} |f(z)| = \infty$, meaning, given $M>0$, there is a δ with $0<\delta<\rho$ such that $|f(z)|>M$ when $|z-z_0|<\delta$, $z\neq z_0$.

<u>Def^n</u> z_0 is removable means $\exists$ value w_0 such that

$$F(z)=\begin{cases} f(z) & z\in D_\rho(z_0)-\{z_0\} \\ w_0 & z=z_0 \end{cases}$$

is analytic on $D_\rho(z_0)$.

<u>Def^n</u> z_0 is essential if, for every ε with $0<\varepsilon<\rho$, $f\left(D_\varepsilon(z_0)-\{z_0\}\right)$ is <u>dense</u> in $\mathbb{C}$, meaning that every pt in $\mathbb{C}$ is a limit pt of $f\left(D_\varepsilon(z_0)-\{z_0\}\right)$.

<u>Goursat/Morera</u> removable sing thm: If f is <u>analytic</u> on $\Omega-\{p\}$ and <u>continuous</u> on Ω, then p is removable.

<u>Next time:</u> Riemann removable sing thm. If f is merely <u>bounded</u> near p, then p is removable.