Lecture 14 Isolated singularities, the pt at ∞, Riemann removable sing thm

Goursat/Morera removable singularity thm. Suppose f is continuous on Ω and analytic on $\Omega - \{a\}$, $a \in \Omega$. Then a is a removable sing for f.

Pf: Goursat $\Rightarrow \int_\Delta f\, dz = 0$ for all Δ's in Ω.

Morera $\Rightarrow f$ analytic on Ω. ✓

Riemann removable singularity thm: Suppose f analytic on $D_\rho(a) - \{a\}$ and bounded there (meaning $\exists M > 0$ such that $|f(z)| < M$, z in set). Then a is a removable sin for f.

Way false $\mathbb{R} \to \mathbb{R}$: $\sin \frac{1}{x}$

Pf Define $G(z) = \begin{cases} (z-a)f(z) & z \neq a \\ 0 & z = a \end{cases}$

$\delta = \frac{\varepsilon}{M}$

f bndd $\Rightarrow$ G continuous on $D_\rho(a)$.

Gour/Morera $\Rightarrow$ G analytic on $D_\rho(a)$.

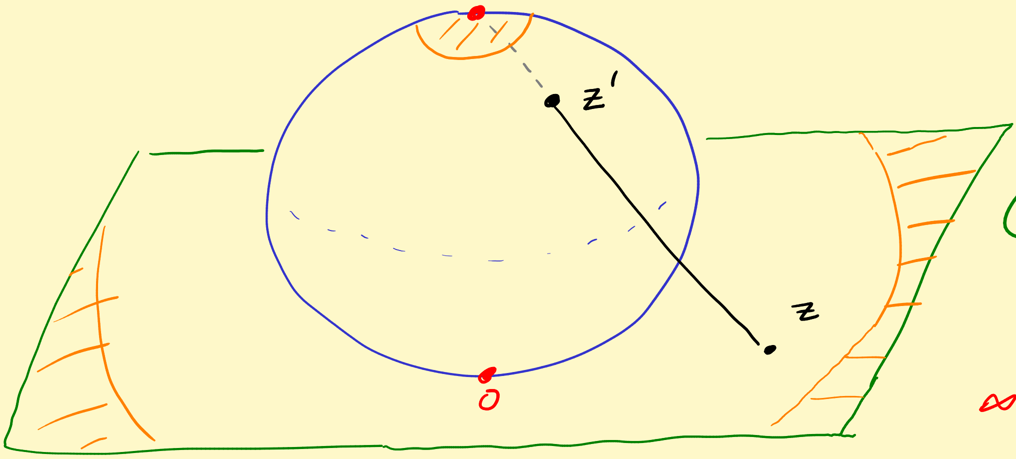
$G(a) = 0 \Rightarrow$ $G(z) = 0 + G'(a)(z-a) + \cdots$

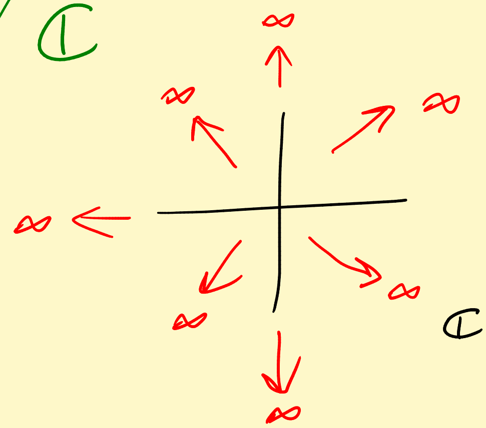
$= (z-a)[a_1 + a_2(z-a) + \cdots] \leftarrow g(z)$

$= (z-a)\underbrace{g(z)}$ Same RoC as G's p.s.

$f(z) = \frac{G(z)}{z-a} = g(z) \leftarrow$ analytic on $D_\rho(a)$. ✓
$z \neq a$

Riemann Sphere





$$D_\varepsilon(\infty) = \{z : |z| > \tfrac{1}{\varepsilon}\} \cup \{\infty\}$$

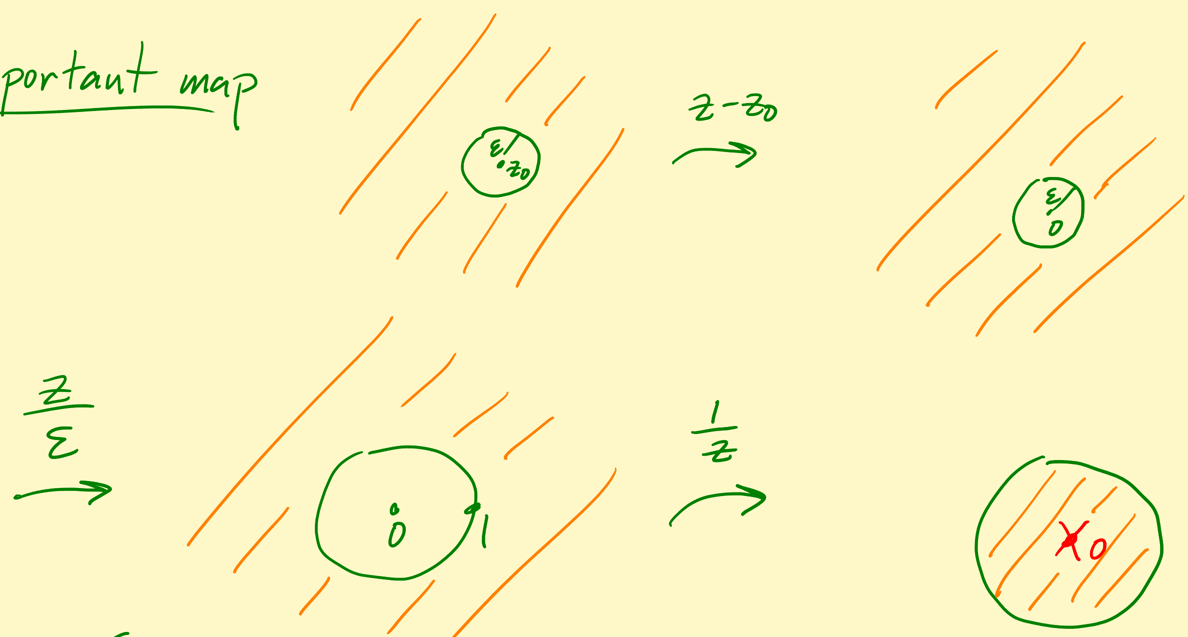
$\lim\limits_{z\to\infty} f(z) = \infty$: Given $M_1 > 0$, $\exists\, M_2 > 0$ such that $|f(z)| > M_1$ when $|z| > M_2$.

Set $f(\infty) = \infty$ makes f cont at ∞ : $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

↖ extended complex plane

$\lim\limits_{z\to z_0} f(z) = \infty$. Setting $f(z_0) = \infty$ makes f cont at z_0 : $\mathbb{C} \to \hat{\mathbb{C}}$.

Important map



$$h(z) = \frac{\varepsilon}{z - z_0}$$

<u>Casorati–Weierstraß thm</u> If an entire fcn misses an open set, it must be constant.

<u>Pf</u> f entire, $f(\mathbb{C}) \cap D_\varepsilon(z_0) = \emptyset$.

Compose with h. Get $h \circ f$ bdd entire!

Liouville $\Rightarrow$ $h(f(z)) \equiv c$, const. Note: $c \neq 0$.

$$\frac{\varepsilon}{f(z) - z_0} = c. \qquad \text{See} \quad f(z) \equiv z_0 + \frac{\varepsilon}{c} \Leftarrow \text{const}$$

<u>Thm</u> Only 3 kinds of isolated sing: Remov, Pole, Ess.

<u>Pf</u> Suppose f analytic on $D_\rho(z_0) - \{z_0\}$.

Suppose z_0 is <u>not</u> essential. [Will show that z_0 is either removable or a pole.]

Then $\exists\ \varepsilon > 0$ with $0 < \varepsilon < \rho$ such that

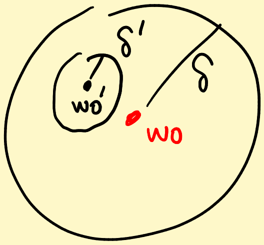
$$\mathcal{A} = f\left(D_\varepsilon(z_0) - \{z_0\}\right) \text{ is } \underline{\text{not}} \text{ dense in } \mathbb{C}.$$

So $\exists\ w_0 \in \mathbb{C}$ that is <u>not</u> a lim pt of $\mathcal{A}$.

<u>Case $w_0 \notin \mathcal{A}$</u> Then $\exists\ \delta > 0$ such that $\underline{D_\delta(w_0) \cap \mathcal{A} = \emptyset}$.

f maps to the <u>outside</u> of $D_\delta(w_0)$

Case $w_0 \in \mathcal{A}$. Then $\exists\ \delta$ such that $D_\delta(w_0) \cap \mathcal{A} = \{w_0\}$.



Aha! Get $\underbrace{D_{\delta'}(w_0')}_{\text{disc missed by } f \text{ on } D_\varepsilon(z_0) - \{z_0\}} \cap \mathcal{A} = \emptyset$

Replace w_0', δ' by w_0, δ notation.

Let $h(z) = \dfrac{\delta}{z - w_0}$.

$h \circ f : D_\varepsilon(z_0) - \{z_0\} \longrightarrow D_1(0) - \{0\}$

is bdd. RRST $\Rightarrow$ z_0 removable.

$$h(f(z)) = \frac{\delta}{f(z) - w_0} = F(z),$$ analytic on $D_\varepsilon(z_0)$.

Solve for f: $\quad f(z) = w_0 + \dfrac{\delta}{F(z)}$

Case $F(z_0) \neq 0$. See z_0 is removable for f.

Case $F(z_0) = 0$. Clear that z_0 is an isolated zero of F.

So $F(z) = (z - z_0)^N g(z)$, $\quad g(z_0) \neq 0$.

$$f(z) = w_0 + \frac{\delta}{(z - z_0)^N} \cdot \frac{1}{g(z)}$$

See z_0 is a pole of f.

Zeroes & poles

Zeroes: $f(z) = (z-a)^N F(z)$

where $F(a) \neq 0$.

Fact 1) f has a zero of order N at $a \iff$

$f(z) = (z-a)^N F(z)$ where F is analytic

near a and $F(a) \neq 0$.

2) f has a pole of order $N \iff$

$f(z) = (z-a)^{-N} F(z)$ where F is analytic

near a and $F(a) \neq 0$.

Why 2 Suppose a is a pole of f. Then

$\lim_{z \to a} f(z) = \infty$. Take $M=1$: $|f(z)| > 1$

when $|z-a| < \delta$, $z \neq a$. So f is nonvanishing

on $D_\delta(a) - \{a\}$. So $h(z) = \frac{1}{f(z)}$ is analytic

on $D_\delta(a) - \{a\}$ and $|h(z)| < 1$ there.

RRST $\Rightarrow$ a is removable for h, and

$f(z) \to \infty$ as $z \to a$ $\Rightarrow$ $h(z) \to 0$ as $z = 0$.

So h has a zero of some order N at a.

$$\frac{1}{f(z)} = (z-a)^N G(z)$$

$$f(z) = (z-a)^{-N} \underbrace{\left[\frac{1}{G(z)} \right]}_{F(z) \text{ analytic near } a. \checkmark}$$

<u>Routine for poles</u> $\quad f(z) = (z-a)^{-N} F(z)$

$$= \frac{1}{(z-a)^N} \left[A_0 + A_1 (z-a) + \cdots + \qquad \right]$$

$$= \underbrace{\frac{A_0^{\neq 0}}{(z-a)^N} + \frac{A_1}{(z-a)^{N-1}} + \cdots + \frac{A_{N-1}}{z-a}}_{\text{Principal part } R(z)} + \underbrace{A_N + \cdots}_{\substack{\text{regular p.s.} \\ \text{with same} \\ \text{RoC as p.s for } F}}$$

<u>Defn</u> $N =$ order of pole

<u>Fact</u> Princ part R is uniquely determined by condition that $f(z) - R(z)$ has <u>removable</u> sing at $z=a$.

<u>Why</u> If R_1, R_2 both do this, then $R_1 - R_2$ has a remove sing. Show $N_1 = N_2$ and all coeff same.

Picard's little theorem: Suppose f is entire. Then f is either a polynomial, or f takes every value in $\mathbb{C}$ infinitely many times with one possible exception (like e^z misses zero).

In particular, an entire fcn that misses two values must be constant.

EX e^z misses 0. $\cos z$ misses nothing.

Picard's Big theorem: If f analytic on $D_\rho(z_0) - \{z_0\}$ and has an essential sing at z_0, f maps $D_\varepsilon(z_0) - \{z_0\}$ ∞-to-one on $\mathbb{C}$ with one possible exception (like $e^{1/z}$ at $z=0$) $(0<\varepsilon<\rho)$