Lecture 15 Singularities at ∞, partial fractions

Def^n If f is analytic on $D_\varepsilon(\infty) - \{\infty\} = \{z : z \in \mathbb{C}, |z| > \frac{1}{\varepsilon}\}$ for some $\varepsilon > 0$, we say f has "an isolated singularity at ∞"

Def^n Type of singularity of $f(z)$ at ∞ is the type of sing of $f(1/z)$ at $z=0$.

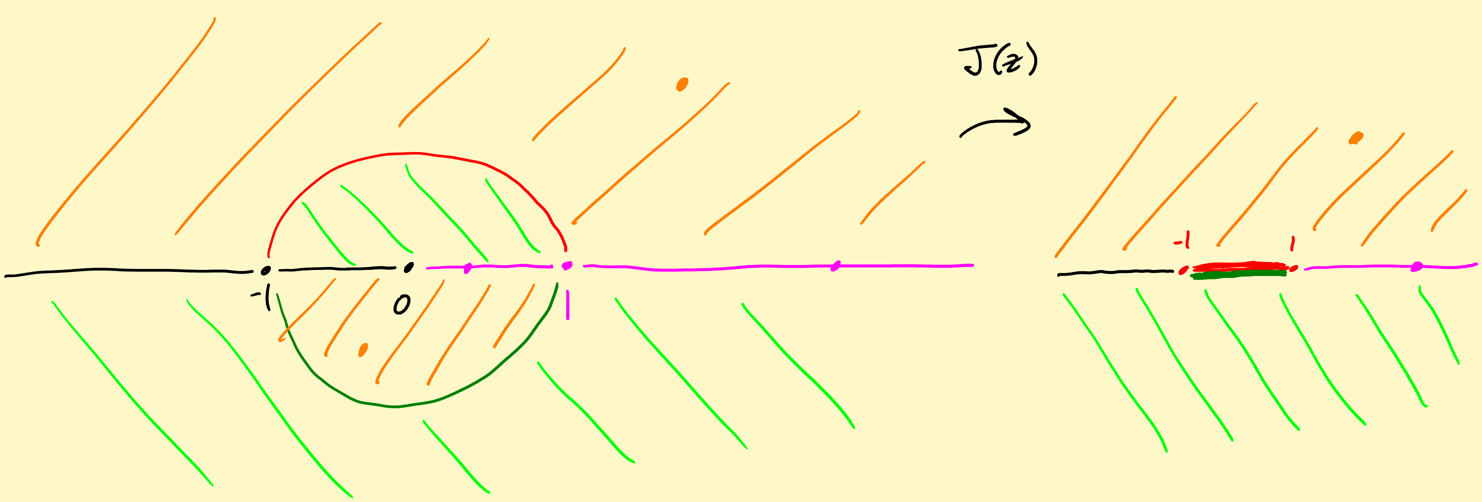
EX e^z has an ess sing at ∞ because $e^{1/z}$ has one at $z=0$.

EX Entire fcns have isolated sing at ∞.

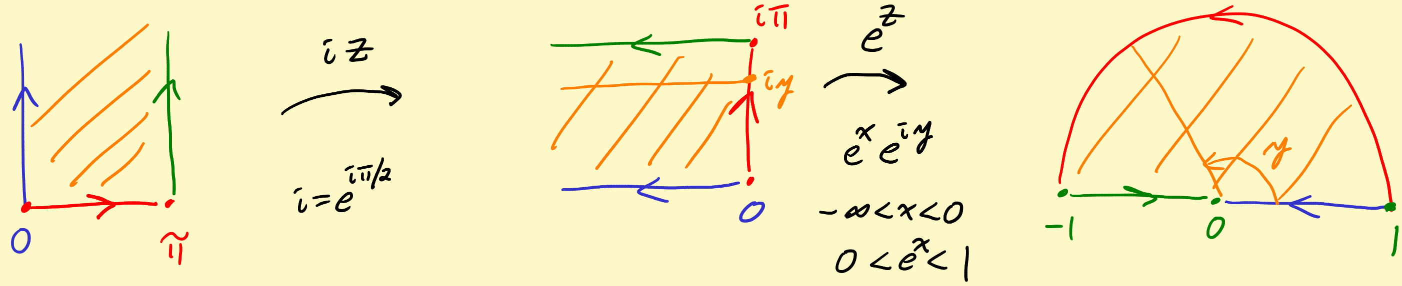
Fact An entire fcn has

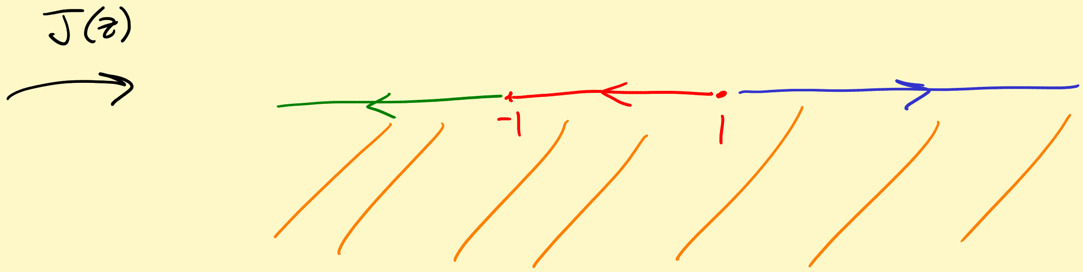
1) a removable sing at ∞ $\Leftrightarrow$ it is constant,
2) a pole at ∞ $\Leftrightarrow$ it is a poly of deg $N \geq 1$. Order of pole = deg of poly,
3) Other possibility is an ess sing at ∞.

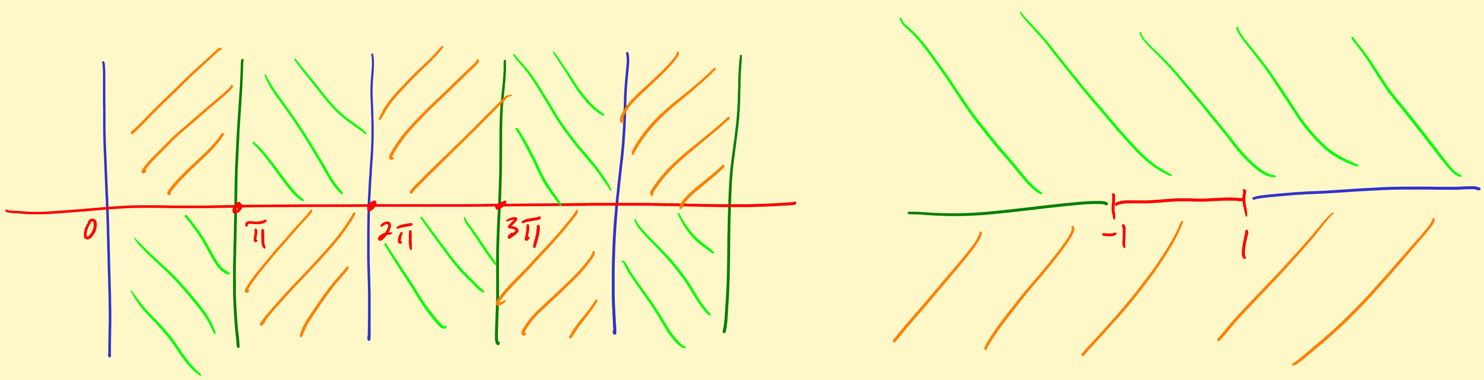
Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$



Complex cosine $\quad \cos z = \frac{1}{2}\left(e^{iz} + \underbrace{e^{-iz}}_{=1/e^{iz}}\right) \quad = J\left(e^{iz}\right)$







$\cos(-z) = \cos z$

$\cos(z + \pi) = -\cos z$

$\cos : \mathbb{C} \to \mathbb{C} \quad \infty\text{-to-one}$

Same for Sin

$\sin z = \cos\left(z - \frac{\pi}{2}\right)$

<u>Partial fractions</u> Suppose $\underbrace{P, Q \text{ polys}}_{\text{with no common factors}}$ $N_P = \deg P < N_Q = \deg Q$.

$\dfrac{P(z)}{Q(z)}$ has a partial fractions decomp as follows:

$$Q(z) = A\,(z - r_1)^{m_1}(z - r_2)^{m_2} \cdots (z - r_n)^{m_n}$$

where $\sum_{k=1}^{n} m_k = N_Q$.

Near r_k: $$\frac{P(z)}{Q(z)} = \frac{1}{(z-r_k)^{m_k}}\left[\frac{P(z)}{q_k(z)}\right]$$

assume no common factors P, Q. $P \neq 0$ at r_k too.

q_k poly, $\neq 0$ at r_k.

$$\underbrace{}_{A_0 + A_1(z-r_k) + A_2(z-r_k)^2 + \cdots}$$

$$= \underbrace{\frac{A_0}{(z-r_k)^{m_k}} + \cdots + \frac{A_{m_k-1}}{z-r_k}}_{R_k(z)} + \left[\text{power series in } (z-r_k)\right]$$

princ. part. (pole order m_k.)

Claim: $$\frac{P}{Q} = \sum_{k=1}^{n} R_k(z)$$ ← part frac decomp

Pf Let $$f(z) = \frac{P}{Q} - \sum_{k=1}^{n} R_k(z)$$

claim f has removable sing at each r_k.

Near r_{k_0}: $$f(z) = \left[R_{k_0}(z) - \text{p.s}\right] - \sum_{k=1}^{n} R_k(z)$$

$$= \underbrace{(\text{p.s.})}_{\text{analytic near } r_k} - \sum_{k \neq k_0} \underbrace{R_k(z)}_{\text{analytic near } r_k.}$$

See r_{k_0} removable.

Give f values at r_k's to make it entire.

Claim $\lim_{z\to\infty} f(z) = 0.$

Clear that $R_k(z) \to 0$ as $z\to\infty$: $\frac{A}{(z-r)^m} \xrightarrow[z\to\infty]{} 0$

$$A|z|^{N_P} \le \underline{|P(z)| \le B|z|^{N_P}}, \quad |z| > R_P$$

$$\underline{a|z|^{N_Q} \le |Q(z)|} \le b|z|^{N_Q}, \quad |z| > R_Q$$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{B}{a}\frac{1}{|z|^{N_Q-N_P}}, \quad |z| > \mathrm{Max}(R_P, R_Q)$$

$\to 0$ as $z\to\infty$ because $N_Q - N_P \ge 1$.

f bndd entire. Liouville's $\Rightarrow$ $f \equiv$ const.

$f(z) \to 0$ as $z\to\infty$ $\Rightarrow$ const $= 0$.

Freshman calc

$$\frac{x^5+x^3+x+1}{(x-1)(x-2)^2\underbrace{(x^2+2x+2)}_{\text{conj pair complex roots}}}$$

$$= \frac{A}{x-1} + \underbrace{\frac{B}{(x-2)^2} + \frac{C}{(x-2)}} + \underbrace{\frac{Dx+E}{(x^2+2x+2)}}_{\frac{F}{x-r} + \frac{\bar{F}}{x-\bar{r}}}$$

Schwarz lemma Suppose f is analytic on $D_1(0)$ and $f: D_1(0) \circlearrowleft$ with $f(0)=0$.

Then A) $|f(z)| \le |z|$ for $z \in D_1(0)$

B) $|f'(0)| \le 1$

and if equality occurs in (A) for some $z \neq 0$

or if equality holds in (B), then

$$f(z) = \lambda z \text{ for some unimodular const } \lambda.$$

Think $f(z) = e^{i\theta} \cdot z$ ← rotation by angle θ