Lecture 16 The Schwarz lemma, log and roots, the Argument principle

HWK 4 due Thurs, Practice problems on home page, Exam 1 next Wed in class.

Schwarz lemma Suppose f is analytic on $D_1(0)$ and $f: D_1(0) \circlearrowleft$ with $f(0)=0$.

Then A) $|f(z)| \le |z|$ for $z \in D_1(0)$

B) $|f'(0)| \le 1$

and if equality occurs in (A) for some $z \ne 0$

or if equality holds in (B), then

$f(z) = \lambda z$ for some unimodular const λ.

$\lambda = e^{i\theta}$: $f(z)$ rotation by θ

Pf

$$F(z) = \begin{cases} \dfrac{f(z)}{z} & z \ne 0 \\ f'(0) & z = 0 \end{cases} \qquad \frac{f(z)}{z} = \frac{f(z)-f(0)}{z-0}$$

F is cont at $z=0$. R.R.S.T. $\Rightarrow$ 0 is removable sing.

Let $z_0 \in D_1(0)$. Pick radius r with $|z_0| < r < 1$.

Max Princ : $|F(z_0)| \le \max_{|z| \le r} |F(z)| = \max_{|z|=r} |F(z)|$

$|z|=r$: $\quad |F(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} < \frac{1}{|z|} = \frac{1}{r}$ when $|z|=r$.

Aha! $|F(z_0)| < \frac{1}{r}$, $\quad \frac{1}{r} \searrow 1$ as $r \nearrow 1$.

Let $r \nearrow 1$. Get $|F(z_0)| \leq 1$.

z_0 arbitrary. So $|F(z)| \leq 1 \implies$ (A), (B).

Part 2 : $=$ in (A) for $z_0 \neq 0$: $|F(z_0)| = 1$

$=$ in (B) : $|F(0)| = 1$

See $|F(z)|$ assumes a local max at $z_0 \in D_1(0)$.

Max Princ $\implies$ $F(z) \equiv \text{const} \leftarrow = F(z_0) \leftarrow$ unimodular const λ

$z \neq 0$: $F(z) = \frac{f(z)}{z} = \lambda$: $\quad \underline{f(z) = \lambda z}$

true at $z=0$ too. ✓

Logarithms and roots on a convex open set (or any domain where the Cauchy thm holds)

Think Given non-vanishing analytic fcn $F(z)$ on convex open domain Ω. Want analytic G with $F(z) = e^{G(z)}$.

Hmmm. If we had G : $\quad \frac{F'(z)}{F(z)} = \frac{e^{G(z)}\,G'(z)}{e^{G(z)}}$

Aha! $\quad G'(z) = \frac{F'(z)}{F(z)}$

Candidate for G : Antiderivative for $\frac{F'}{F}$.

Pf Define $$G(z) = \int_{L_a^z} \frac{F'(w)}{F(w)}\,dw$$ $\leftarrow$ a fixed

Know $G' = \frac{F'}{F}$

Is $e^G = F$? Almost!

$$\frac{d}{dz}\left(\frac{F}{e^G}\right) = \frac{F' e^G - F\cdot[e^G G']}{(e^G)^2} \equiv 0$$

$G' = \frac{F'}{F}$

So $\frac{F}{e^G} \equiv c$, a const, on Ω.

Let $b \in \log c$ $(c \neq 0)$

$$F = c e^G = e^b e^G = e^{b+G}$$ $\leftarrow$ complex analytic $\log F$

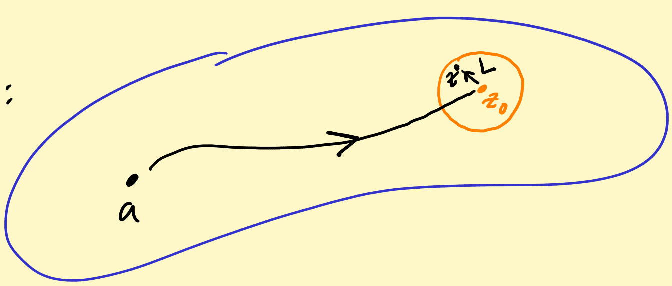
Ω domain with Cauchy thm:

"Define" $$G(z) = \int_{\gamma_a^z} \frac{F'(w)}{F(w)}\,dw$$

Step 1: Well defined: $\gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is closed

Cauchy $\Rightarrow$ $G(z) = \tilde{G}(z)$.

Step 2:



$$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$$

Step 3 Get $G' = \frac{F'}{F}$ on $D_\rho(z_0)$.

Step 4 $F = e^{b+G}$ on $D_\rho(z_0)$.

Extends to hold on domain Ω by Identity thm.

Fact Analytic log fcns give rise to analytic root fcns.

Why: $F(z)$ non-vanishing analytic

$$F(z) = e^{G(z)}$$

Aha! $$F(z) = \Big[\underbrace{e^{G(z)/N}}_{\text{analytic } N\text{-th root of } F}\Big]^N$$

Def^n If z_0 is a pole of $f(z)$, we know

$$f(z) = (z-z_0)^{-N}\,F(z), \quad F \text{ analytic}, \ F(z_0) \neq 0$$

$$= \underbrace{\left[\frac{A_0}{(z-z_0)^N} + \cdots + \frac{A_{N-1}}{z-z_0}\right]}_{\text{princ part at } z_0} + (\text{regular power series})$$

A_{N-1} is the "residue of f at z_0".

Write $A_{N-1} = \mathrm{Res}_{z_0} f$

Fact: At a $\left\{\begin{array}{l}\text{zero of order } N \\ \text{pole of order } N\end{array}\right\}$, the residue

of $\dfrac{f'(z)}{f(z)}$ is $\left\{\begin{array}{c} N \\ -N \end{array}\right\}$.

Pf $\quad f(z) = (z-z_0)^N F(z) \qquad \left\{\begin{array}{ll} N>0 & \text{zero} \\ N<0 & \text{pole}\end{array}\right.$

$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1}F(z) + (z-z_0)^N F'(z)}{(z-z_0)^N F(z)}$$

$$\frac{f'(z)}{f(z)} = \underbrace{\frac{\overset{\text{Residue}}{N}}{(z-z_0)}}_{\text{princ. part}} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic}} \quad F(z_0) \neq 0$$

Argument principle for a disc. Suppose f analytic on $D_R(z_0)$ and $0<r<R$ and f has no zeroes on $C_r(z_0) = \{z : |z-z_0| = r\}$. Then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity}\end{array}\right)$$

Pf Note: f can only have finitely many zeroes $z_1, z_2, \dots, z_N$ in $D_r(z_0)$. [$\overline{D_r(z_0)}$ is compact. ∞ many $\Rightarrow$ they have a limit pt. lim pt can't be on $C_r(z_0)$ (lim pts of zeroes are zeroes for cont fcns). Identity thm $\Rightarrow f \equiv 0$ $\lightning$]

Say z_k is a zero of mult m_k.

Now $\dfrac{f'}{f} - \displaystyle\sum_{k=1}^{N} \dfrac{m_k}{z-z_k}$ has removable sing at each z_k. So Cauchy thm yields

$$0 = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz - \sum_{k=1}^{N} \frac{m_k}{2\pi i} \underbrace{\int_{C_r(z_0)} \frac{1}{z-z_k}\,dz}_{2\pi i}$$