Lecture 17 Open mapping theorem & consequences

Exam 1 in class on Wed
Review on Monday

Open mapping theorem Suppose f is a non-constant analytic fcn on a domain Ω. Then f maps open sets to open sets. In particular, $f(\Omega)$ is open. (OMT)

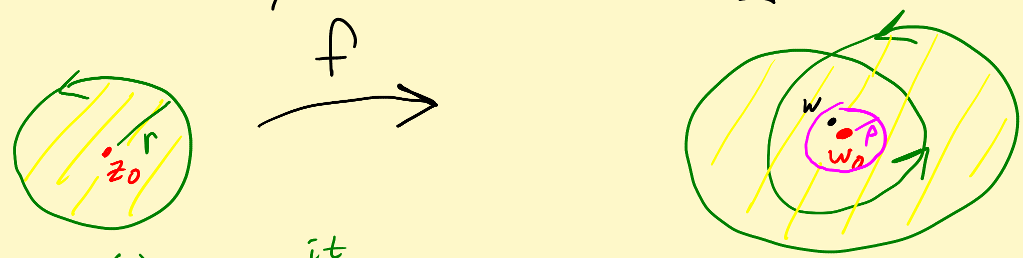
Way false $\mathbb{R} \to \mathbb{R}$ $\quad h(t) = 1 - t^2 \quad h((-1,1)) = (0,1]$

Notation: $f : \Omega \to \mathbb{C}$. $\quad f(U) = \{f(z) : z \in U\}$

$f^{-1}(U) = \{z \in \Omega : f(z) \in U\}$

Fact f continuous $\iff$ $f^{-1}(U)$ is open whenever U is open.

Pf that Arg princ $\Rightarrow$ OMT. f not constant, analytic on Ω. Want to see $f(\Omega)$ is open. Suppose $w_0 \in f(\Omega)$. Then $\exists\, z_0 \in \Omega$ with $f(z_0) = w_0$. Note that $f(z) - w_0$ is not constant, so zero at z_0 is isolated. So $\exists\, r > 0$ so that $\overline{D_r(z_0)} \subset \Omega$ and z_0 is the only zero of $f(z) - w_0$ in $\overline{D_r(z_0)}$. Let N = order of zero of $f(z) - w_0$ at z_0. [We say that $f(z) = w_0$ with multiplicity N at $z = z_0$.]



$C_r(z_0)$: $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$f(C_r(z_0))$: $f(z(t))$, $0 \le t \le 2\pi$

$|f(z(t)) - w_0|$ is non-vanishing cont fcn on $[0, 2\pi]$,

so it has a positive min $m > 0$. Pick ρ with $0 < \rho < m$

Claim: $D_\rho(w_0) \subset f(D_r(z_0))$. Key: Arg princ

$$\underbrace{\begin{pmatrix} \#\text{ times } f \text{ "hits" } w \\ \text{in } C_r(z_0) \end{pmatrix}}_{\substack{\#\text{ zeroes of } f(z)-w \\ \text{counted with mult}}} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{\underbrace{f(z) - w}} \, dz = H(w)$$

$F(z) = f(z) - w$

$F'(z) = f'(z)$

Notice that $H(w_0) = N$. We just want to see that

$H(w) \geq 1$ when $w \in D_\rho(w_0)$.

Claim $H(w)$ is analytic on $D_\rho(w_0)$, and so an integer valued continuous fcn on connected set $D_\rho(w_0)$, and so $\equiv N$.

$$H(w) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z) - w} \, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(z(t))\, z'(t)}{\underbrace{f(z(t))}_{!} - w} \, dt \qquad \frac{d}{dt} f(z(t))$$

$$= \frac{1}{2\pi i} \int_{f(C_r(z_0))} \frac{1}{\zeta - w} \, d\zeta$$

HWK 2: prob 1 Analytic in w. ✓

$$= \mathrm{Ind}_{f(C_r(z_0))}(w)$$ ← Index or winding # of curve $f(C_r(z_0))$ in the counterclockwis direction about w.

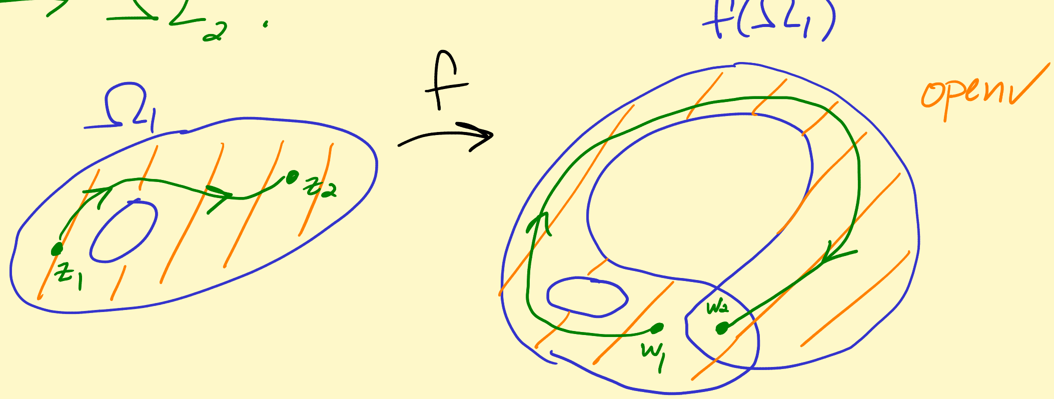
Done ✓ $D_\rho(w_0) \subset f(\Omega)$. w_0 arbitrary. So $f(\Omega)$ open.

<u>Cor</u> f analytic, non-constant on a domain Ω_1.

Let $\Omega_2 = f(\Omega_1)$. Then Ω_2 is a <u>domain</u> too.

$f : \Omega_1 \xrightarrow{\text{onto}} \Omega_2$.

<u>Pf</u> OMT



<u>Cor OMT</u> f analytic on domain Ω_1 and <u>one-to-one</u>.

Let $\Omega_2 = f(\Omega_1)$. $f : \Omega_1 \underset{\text{onto}}{\xrightarrow{1-1}} \Omega_2$ ← domain. one-to-one ⟹ not const.

Let $F = f^{-1} : \Omega_2 \to \Omega_1$. Note that

OMT ⟹ F is continuous. $\left[F^{-1}(U) = f(U) \text{ open if } U \text{ is.} ✓ \right]$

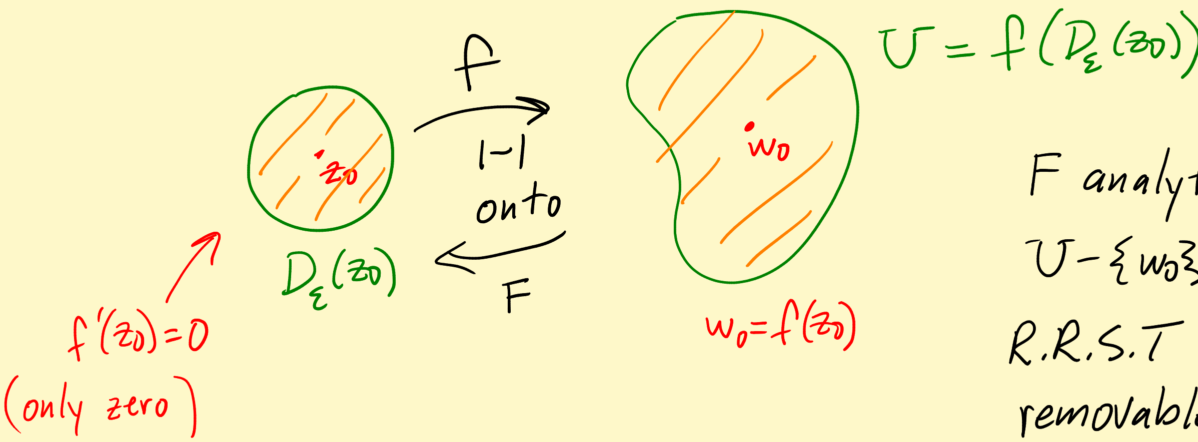
1) F is <u>analytic</u> on Ω_2, and

2) $F'(f(z)) = \dfrac{1}{f'(z)}$ ← $f'(z)$ is non-vanishing on Ω_1.

Way false $\mathbb{R} \to \mathbb{R}$. $h(t) = t^3$. $h : (-1,1) \xrightarrow[\text{onto}]{1-1} (-1,1)$

$h^{-1}(\tau) = \tau^{1/3}$ $\leftarrow$ not diff'ble at $t=0$.

Pf f not const. So f' is not $\equiv 0$. Zeroes of f' are isolated. Suppose $f'(z_0) = 0$.



F analytic on $U - \{w_0\}$, Cont at w_0. R.R.S.T $\Rightarrow$ w_0 is removable.

will prove properly after Exam 1

Hmmm. $F(f(z)) = z$

$$F'(f(z))\,f'(z) = 1$$

Aha! $f'(z)$ can't vanish!

After Exam 1 : Inverse fcn thm for analytic fcns.