Review for Exam 1

Exm 1 in class Wed, Oct 2

3. (25 pts) Let C_R denote the semicircle parameterized by $z(t) = Re^{it}$, $0 \le t \le \pi$. Prove that

$$I = \int_{C_R} \frac{z^3 + z^2 + z + 1}{z^5 + z^3 + z + 1}\, dz$$

$\leftarrow P(z)$

$\leftarrow Q(z)$

tends to zero as $R \to \infty$.

$R >$ radius of circ containing all the zeroes of denom $\leftarrow R_z$

P or Q

$$\underbrace{a|z|^N}_{Q \text{ side}} \le |\ (z)| \le \underbrace{A|z|^N}_{P \text{ side}}, \quad |z| > R_0.$$

$$|I| \le \underbrace{\left(\max_{|z|=R} \left|\frac{P(z)}{Q(z)}\right|\right)}\, 2\pi R$$

$$\le \frac{AR^3}{aR^5} \cdot 2\pi R$$

$R > \operatorname{Max}(R_P, R_Q, R_z)$

$\longrightarrow 0$ as $R \to \infty$. ✓

4. (25 pts) Suppose f is analytic on $D_1(0) - \{0\}$ and there are real constants M and λ with $M > 0$ and $0 < \lambda < 1$ such that

$$|f(z)| \leq \frac{M}{|z|^\lambda}$$

on $D_1(0) - \{0\}$. Prove that f must have a removable singularity at $z = 0$.
Hint: Consider $zf(z)$.

Important theorems: Liouville's, Cauchy estimates, Fund Thm Alg, Max Princ, Schwarz lemma, Riemann R.S.T.

Hmmm. $|zf(z)| \leq |z| \frac{M}{|z|^\lambda} = M|z|^{1-\lambda}$ $\leftarrow >0!$

$\longrightarrow 0$ as $z \to 0$.

So $zf(z)$ has a removable sing at 0 by RRST

and it is zero at $z=0$.

$$F(z) = \begin{cases} zf(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$$ is analytic on $D_1(0)$.

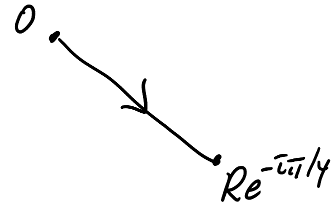
Taylor series : $F(z) = \underbrace{F(0)}_{0} + a_1 z + a_2 z^2 + \cdots$

$$= z \underbrace{\left[a_1 + a_2 z + \cdots \right]}_{\text{R of C} \geq 1}$$

$= z\tilde{f}(z)$, $\tilde{f}$ analytic on $D_1(0)$.

Aha! $zf(z) = z\tilde{f}(z)$, $z \neq 0$. So $f(z) = \tilde{f}(z)$, $z \neq 0$.
Define $f(0) = a_1 = F'(0)$.

2. (25 pts) Let $R > 0$ and let L denote the path that starts at $z = 0$ and follows the line from zero to $Re^{-i\pi/4}$ parameterized by $z(t) = te^{-i\pi/4}$, $0 \le t \le R$. Express the integral $\int_L e^{-z^2}\,dz$ as $a + bi$ where a and b are real integrals of real valued functions.



$z'(t) = e^{-i\pi/4}$

$$\int_L e^{-z^2}dz = \int_0^R e^{-\left(t e^{-i\pi/4}\right)^2} e^{-i\pi/4}\, dt \qquad \left(e^{-i\pi/4}\right)^2 = e^{-i\pi/2} = -i$$

$$= e^{-i\pi/4}\int_0^R \underbrace{e^{it^2}}_{\cos t^2 + i \sin t^2} dt$$

$$= \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\left[\int_0^R \cos t^2\, dt + i\int_0^R \sin t^2\, dt\right]$$

3. Suppose that f is an analytic function on the unit disc such that $|f(z)| < 1$ for $|z| < 1$. Prove that if f has a zero of order n at the origin, then

$$|f(z)| \le |z|^n$$

for $|z| < 1$. How big can $|f^{(n)}(0)|$ be? $\le n!$

Part 2: $f(z) = \lambda z^n$, $|\lambda| = 1$.

Redo pf of Schwarz $\quad f(z) = z^n F(z)$, F analytic. $F(0) \neq 0$.

$$F(0) = \frac{f^{(n)}(0)}{n!}$$

$$H(z) = \begin{cases} \dfrac{f(z)}{z^n} & z \neq 0 \\ F(0) & z = 0 \end{cases}$$

Pick $z_0 \in D_1(0)$. Let $r > 0$ be such that $0 < |z_0| < r < 1$.

$$|H(z_0)| \le \max_{|z|=r} \left| \frac{f(z)}{z^n} \right| < \frac{1}{r^n} \searrow 1$$

as $r \nearrow 1$. So $|H(z_0)| \le 1$.

2. Given a complex polynomial $P(z)$ of degree $N \geq 1$, there exist real constants $0 < a < A$ and $R > 0$ such that the *basic polynomial estimate*,

$$a|z|^N \leq |P(z)| \leq A|z|^N$$

holds for $|z| > R$. Show that, if an entire function $f(z)$ satisfies the right hand side of the basic polynomial estimate, it must be a polynomial of degree N or less. Show that if an entire function satisfies the left hand side, it must be a polynomial of degree N or more.

Left: Fact: f entire and $|f(z)| \to \infty$ as $|z| \to \infty$, then f is poly.

Hmmm. $\frac{1}{f} \to 0$ as $|z| \to \infty$.

Lemma: f has finitely many zeroes in $\mathbb{C}$.

Pf: $|f(z)| > 1$ for $|z| > R$.

If f had ∞ many zeroes in compact $\overline{D_R(0)}$,

they'd have a lim pt. $\Rightarrow f \equiv 0$ $\lightning$

Partial fractions! $\displaystyle \frac{1}{f(z)} - \sum_{k=1}^{m} \underbrace{R_k(z)}_{} \equiv 0$

Principal parts of $\frac{1}{f}$ at zeroes of f.

Hmmm. $f = \text{Rational} = \dfrac{P(z)}{Q(z)}$ no common zeroes.

Q can't have any zeroes because f is entire.

Last thing to show: $\deg f \geq N$.

Use Basic poly est.

Right. $f(z) = \sum_0^\infty a_n z^n \qquad a_n = \frac{f^{(n)}(0)}{n!}$

Cauchy estimates $\quad |f^{(n)}(0)| \le \frac{n!M}{R^n}$

Take $n > N$. $\quad R \nearrow \infty$.

See $a_n = 0, \quad n > N$.

4. Show that if f is an analytic mapping of the unit disk into itself such that $f(a) = 0$, then

$$|f(z)| \le \left| \frac{z-a}{\underbrace{1-\bar{a}z}} \right|$$

Hwk 1; prob 1

for all z in the disk.

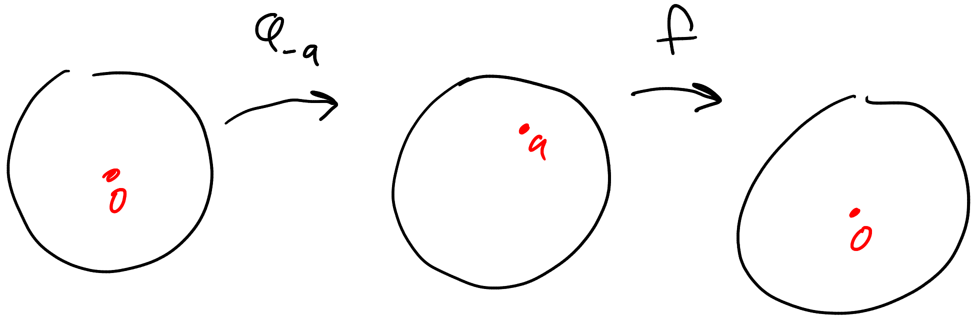
$\varphi_a(z)$ has a simple zero at a.

Could repeat pf of Schwarz using Max princ on

$\dfrac{f(z)}{\varphi_a(z)}$; has a removable sing at $z=a$, etc.

Or

Need $0 \mapsto 0$



Schwarz $\Rightarrow$ $\left| f(\underbrace{\varphi_{-a}(z)}_{w}) \right| \le |z|$ $\quad z = \varphi_a(w)$ ✓

Know $(\varphi_a)^{-1} = \varphi_{-a}$ prob 1.

5. Prove that if h_1 and h_2 are two analytic functions on a domain Ω such that $h_1^N \equiv h_2^N$ for some positive integer N, then there is an N-th root of unity λ such that $h_1 = \lambda h_2$ on Ω.

Hints: Case $h_1 \equiv 0$. Then $h_2 \equiv 0$.

Case: zeroes of h_1, if any, isolated.

Take $z_0 \in \Omega$ and a seq $z_n \to z_0$, $z_n \neq z_0$, $z_n \in \Omega$.

Show: For $n > M$, $\exists$ N-th root of unity μ_n such that $h_1(z_n) = \mu_n h_2(z_n)$.

Hints: Only finitely many possible values of μ_n. Identity thm.

6. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.

Only case to work on. If z_0 is a pole of f, it cannot be a pole of $e^{f(z)} = F(z)$

If it were a pole of order N,

$$-N = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{F'(z)}{F(z)}\, dz = \frac{1}{2\pi i} \int_{C_R(z_0)} f'(z)\, dz$$

$$= 0.$$

Fund Thm Calc.

<u>or</u> show, if f has a pole at z_0, there is an $\varepsilon > 0$ and $R > 0$ such that f maps $D_\varepsilon(z_0) - \{z_0\}$ <u>onto</u> $\{z : |z| > R\}$.

[Hint: Consider $\frac{1}{f(z)}$. It has a zero at z_0.]

Use the graph of e^z I drew in class to deduce that image of $D_\varepsilon(z_0) - \{z_0\}$ under $e^{f(z)}$ is dense in $\mathbb{C}$.

1. What is the radius of convergence of the power series centered at zero for the function $1/(z-1-i)^{10}$?

R of C $\geq$ biggest radius R such that fcn is analytic on $D_R(0)$. So $R \geq |1+i|$.

Pole $\Rightarrow$ $R \leq |1+i|$ too. So $R = \sqrt{2}$.