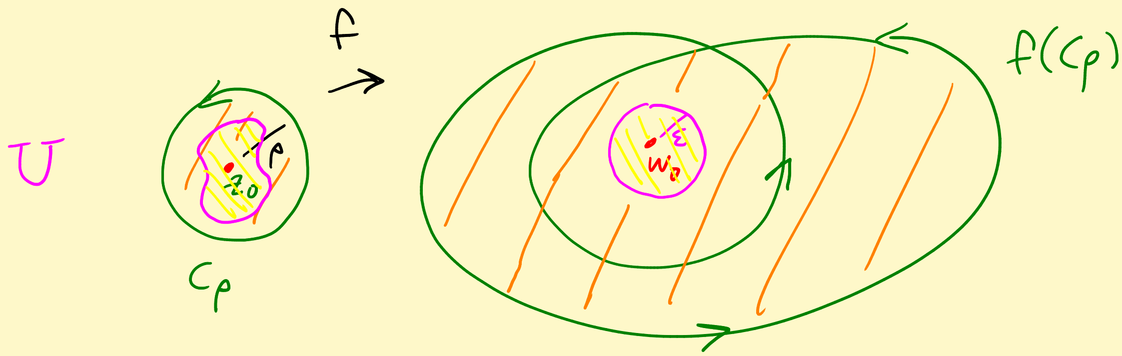
Lecture 18 Inverse function theorem, Local mapping thm

HWK 5 due Thurs, 10/17

Inverse function thm Suppose f is analytic on $D_R(z_0)$ and $f'(z_0) \neq 0$. Say $f(z_0) = w_0$. Then there exists an $\varepsilon > 0$ and a domain $U \subset D_R(z_0)$ such that $z_0 \in U$, $f : U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(w_0)$, and f^{-1} is analytic on $D_\varepsilon(w_0)$ and $(f^{-1})'(w) = \dfrac{1}{f'(f^{-1}(w))}$.

Pf



From pf of OMT : $H(w) = \dfrac{1}{2\pi i} \displaystyle\int_{C_\rho} \frac{f'(z)}{f(z) - w}\, dz =$ # times w gets hit by f on $D_\rho(z_0)$

Showed $H(w) \equiv H(w_0) = N$, order of zero of $f(z) - w_0$ at z_0.

Today: If $f'(z_0) \neq 0$, then $f(z) - w_0$ has a <u>simple</u> zero at z_0, i.e., $N = 1$. So every w in $D_\varepsilon(w_0)$ gets hit exactly once! So f^{-1} exists on $D_\varepsilon(w_0)$. OMT $\Rightarrow$ f^{-1} continuous there. Let $U = f^{-1}(D_\varepsilon(w_0))$. U is a domain, $z_0 \in U$, and f is 1-1 on U.

The rest is easy. [Knowing f^{-1} exists and is cont is the hard part in the real case.]

$w = f(z)$ $\qquad F = f^{-1}$ $\qquad z = F(w)$

$w_0 = f(z_0)$ $\qquad\qquad z_0 = F(w_0)$

F cont $\Longrightarrow$ $\overbrace{F(w)}^{z} \longrightarrow \overbrace{F(w_0)}^{z_0}$ as $w \to w_0$.

$$DQ = \frac{F(w)-F(w_0)}{w-w_0} = \frac{z-z_0}{f(z)-f(z_0)}$$

Aha! We know that $\overset{F(w)}{\overset{\|}{z}} \to \overset{F(w_0)}{\overset{\|}{z_0}}$ as $w \to w_0$

So $DQ = \dfrac{1}{\left(\dfrac{f(z)-f(z_0)}{z-z_0}\right)} \longrightarrow \dfrac{1}{f'(z_0)}$ as $w \to w_0$.

$$F'(w_0) = \frac{1}{f'(z_0)} \quad ✓$$

$$\boxed{\begin{aligned} F(f(z)) &= z \\ F'(f(z))f'(z) &= 1 \end{aligned}} \quad ✓$$

<u>Remark</u> Now the proof of the "Special inverse fcn thm" at the end of Lecture 17 is complete:

f analytic and locally 1-1 $\Longrightarrow$ f' non-vanishing.

<u>Local mapping theorem</u> Say $f(z_0) = w_0$ with mult N, meaning that $f(z) - w_0$ has a <u>zero</u> of order N at $z = z_0$.

<u>Step 1</u> Get disc $D_\rho(z_0)$ so that z_0 is <u>only</u> zero of $f(z) - w_0$ in $\overline{D_\rho(z_0)}$.

<u>Step 2</u> $f(z) - w_0 = (z-z_0)^N F(z)$ where F is analytic

on $D_\rho(z_0)$ and is non-vanishing on $\overline{D_\rho(z_0)}$ (including at z_0).

Step 3 $D_\rho(z_0)$ is convex open, F non-vanishing. So there is an analytic N-th root fcn $g(z)$ with $g(z)^N = F(z)$. g is non-vanishing too.

Step 4

$$f(z) - w_0 = (z - z_0)^N \underbrace{F(z)}_{g(z)^N}$$

$$f(z) = w_0 + \Big[\underbrace{(z-z_0)\, g(z)}_{G(z)}\Big]^N$$

Step 5

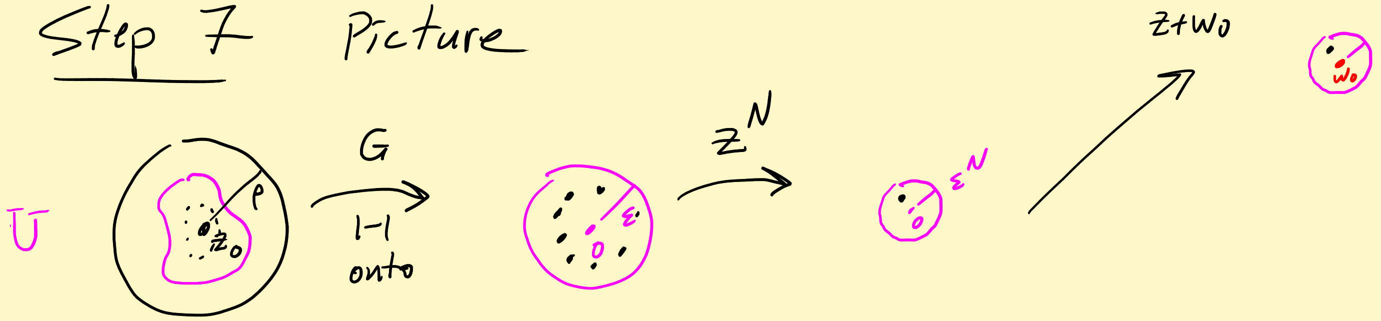
$$G'(z) = 1 \cdot g(z) + (z - z_0) g'(z)$$

$$G'(z_0) = \underbrace{g(z_0)}_{\neq 0} + 0$$

Step 6 Since $G'(z_0) \neq 0$, can apply inverse fcn thm.

Note $G(z_0) = (z_0 - z_0)\, g(z_0) = 0$. Get $D_\varepsilon(0)$ and domain $U \subset D_\rho(z_0)$ such that $G : U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(0)$ non-vanishing derivative.

Step 7 Picture



Understanding z^N is all you need!

See $f(z_0) = w_0$. $\quad f: U - \{z_0\} \longrightarrow D_{\varepsilon^N}(w_0) - \{w_0\}$

N-to-one

$f^{-1}(w) = \{N$ <u>distinct</u> pts$\}$

when $w \neq w_0$.