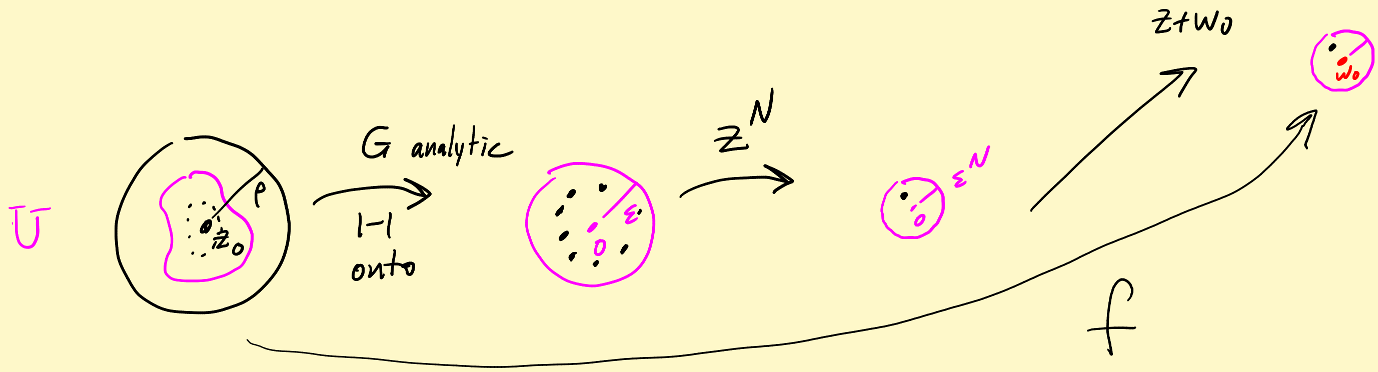
Lecture 19 Conformal mapping, zeroes of harmonic fcns.

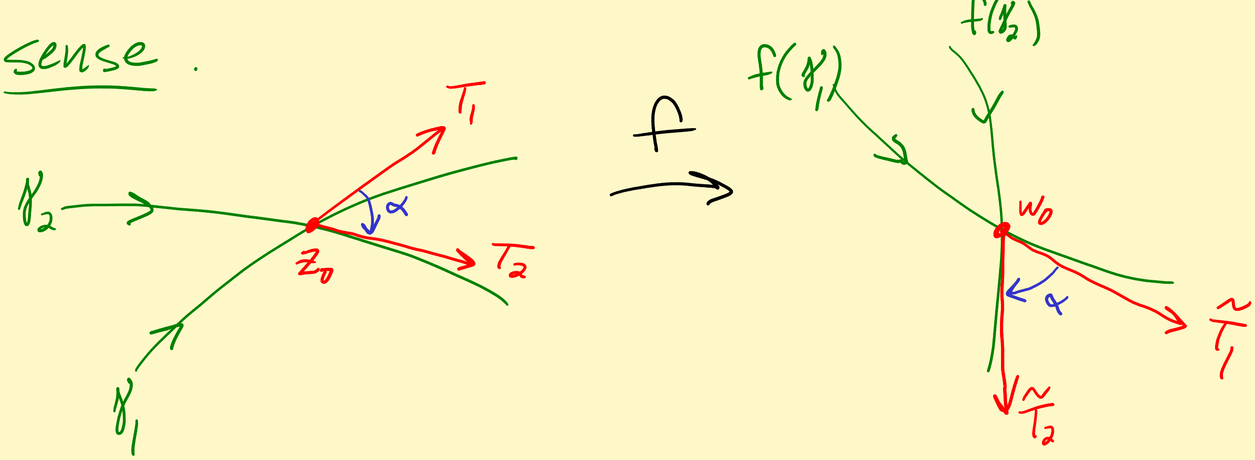
Local mapping theorem: f analytic on $D_\rho(z_0)$, $f(z_0)=w_0$ with multiplicity N [meanig $f(z)-w_0$ has a zero of order N]



$$f(z) = w_0 + \left[G(z) \right]^N$$

Remark Local mapping theorem $\Rightarrow$ f analtic and locally one-to-one [on a small disc about each pt], then f' is non-vanishing $\Leftarrow$ because f 1-1 only happens if $N=1$.

Big fact f analytic, locally 1-1 $\Rightarrow$ f' non-vanishing and conformal, meaning it preserves angles and their sense.



$\gamma_j : z_j(t) \; : \; z_j(0)=z_0, \; j=1,2$ $\qquad$ $f(\gamma_j) : f(z_j(t))$

$$T_j = z_j'(0) \qquad \tilde{T}_j : \frac{d}{dt} f(z_j(t))\Big|_{t=0}$$

$$= f'(\underbrace{z_j(0)}_{z_0})\, z_j'(0)$$

$$= \underbrace{f'(z_0)}_{re^{i\theta}}\, T_j$$

So $\tilde{T}_j = re^{i\theta}\, T_j$ ← stretch by factor of r, rotate by θ (in counterclockwise direction when $\theta > 0$).

See angles, sense are preserved by f.

Fact Partial converse: $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose u, v are C^1-smooth and $\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \neq 0$

If this map is conformal, then $u + iv$ is analytic.

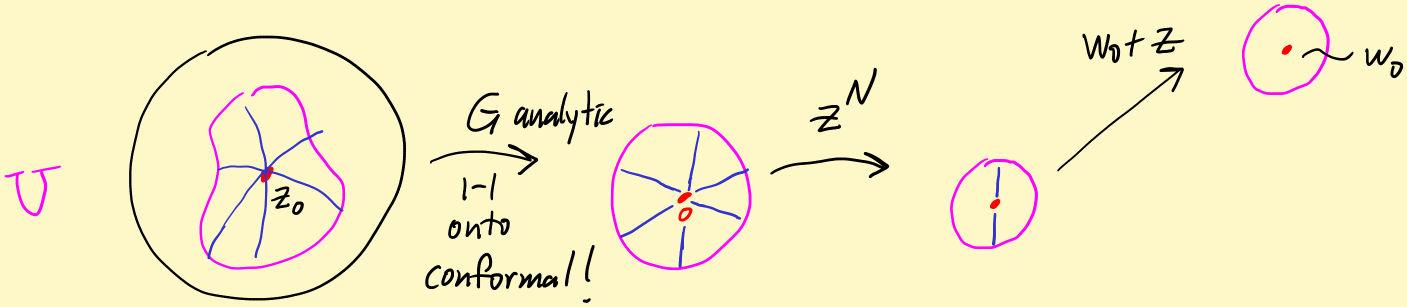
Sketch of pf Jacobian matrix $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ represents a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ that preserves angles.

Analytic geometry: $J = \begin{bmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{bmatrix}$

See C-R eqns!

Local mapping thm gives us a deep understanding of the zeroes of harmonic fcns.

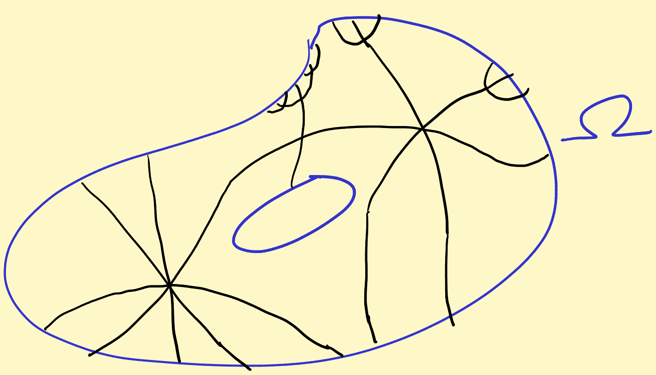
Suppose u is harmonic on a disc $D_\rho(z_0)$, not const.
Let v be a harm conj on $D_\rho(z_0)$. $f = u + iv$.



Suppose $u(z_0) = 0$. Can also suppose $v(z_0) = 0$ [by subtracting const $v(z_0) \neq 0$].

See: zero set of u on U consists of N curves that all cross at z_0, making angles π/N.

Ave princ $\Rightarrow$ zeroes of harmonic fcns cannot be <u>isolated</u>.
This is another way to confirm this.



zero set of harmonic fcn on Ω, a wild spider web!

<u>MA 525</u> Def^n: f analytic: u, v C^1-smooth and satisfy satisfy C-R eqns.

Inverse fcn thm for analytic fcns. $\quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$

$$J = \begin{bmatrix} u_x & \overset{-v_x}{\overset{\|}{u_y}} \\ v_x & \underset{u_x}{\underset{\|}{v_y}} \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

$$\det J = (u_x)^2 + (v_x)^2 = |f'|^2$$

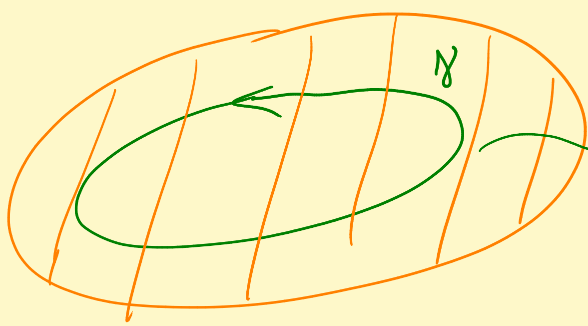
$f'(z_0) \neq 0$. $\quad J^{-1} = \dfrac{1}{A^2+B^2} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ ← See C-R equns!

Fact: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\mathbb{R}^2 \to \mathbb{R}^2$ inverse fcn thm $\Rightarrow f^{-1} = U + iV$ is C^1-smooth.

see f^{-1} is 525-analytic.

<u>Cauchy thm</u>



f analytic

Ω = inside γ

$\gamma: z(t),\ a \le t \le b$

$$\int_\gamma f\, dz = \int_a^b \left[u(x(t),y(t)) + i\, v(x(t),y(t)) \right] (\dot{x}(t) + i\dot{y}(t))\, dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})\, dt + i \int_a^b (u\dot{y} + v\dot{x})\, dt$$

$$= \left(\int_\gamma u\,dx - v\,dy \right) + i \left(\int_\gamma u\,dy + v\,dx \right)$$

$$\underset{\text{Green's thm}}{=} \iint_{\Omega} \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\substack{=0 \\ \text{C-R eqn \#2}}} dA + i \iint_{\Omega} \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\substack{=0 \\ \text{C-R eqn \#1}}} dA$$

$$\boxed{\int_{\gamma} P\,dx + Q\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy} \quad \text{Green's}$$