Lecture 20 $\mathrm{Aut}(D_1(0))$, why "Argument principle"

HWK 1 : p.1 $\quad \varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $a \in D_1(0)$, Möbius transformations

$\varphi_a : D_1(0) \circlearrowleft \substack{1-1 \\ \text{onto}}$ analytic.

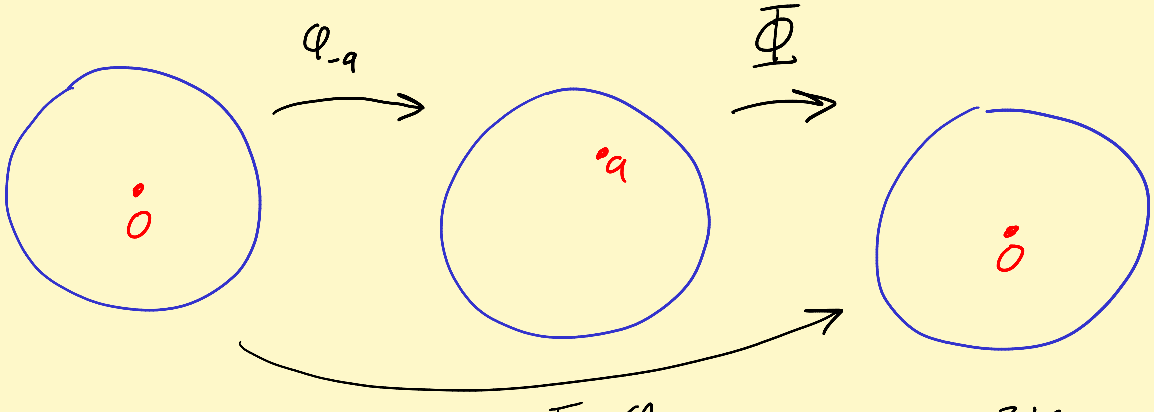
$$\mathrm{Aut}(D_1(0)) = \left\{ \Phi : D_1(0) \xrightarrow[\text{onto}]{1-1} D_1(0) \text{ analytic} \right\}$$

Fact $\quad \mathrm{Aut}(D_1(0))$ is a group under composition

Important reason: f analytic and locally 1-1 $\Rightarrow f'$ non-vanishing. So if $\Phi \in \mathrm{Aut}(D_1(0))$, then Φ^{-1} is analytic and so also in,

Thm : $\mathrm{Aut}(D_1(0)) = \left\{ \lambda \varphi_a : a \in D_1(0),\ |\lambda| = 1 \right\}$

Pf : Suppose $\Phi \in \mathrm{Aut}(D_1(0))$. Let $a = \Phi^{-1}(0)$.



$\Psi = \Phi \circ \varphi_{-a} \qquad \varphi_{-a} = \dfrac{z+a}{1+\bar{a}z}$

(p.1 : Recall $\varphi_a^{-1} = \varphi_{-a}$)

$\Psi : D_1(0) \circlearrowleft \substack{1-1 \\ \text{onto}}$ analytic, and $\Psi(0) = 0$.

Schwarz! $\boxed{|\Psi'(0)| \le 1}$

Aha! Ψ^{-1} analytic and also satisfies $\Psi^{-1}(0) = 0$.

So $\underbrace{|(\Psi^{-1})'(0)|}_{\frac{1}{|\Psi'(\underbrace{\Psi^{-1}(0)}_{0})|}} \le 1$. $\Rightarrow \boxed{|\Psi'(0)| \ge 1.}$

Conclude $|\Psi'(0)| = 1$. Schwarz part 2 says

$\Psi(z) = \lambda z$ for some const λ with $|\lambda| = 1$.

$\Phi(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda z$ $z = (\varphi_{-a})^{-1}(w) = \varphi_a(w)$ ✓

<u>Fact</u> $\mathrm{Aut}(D_1(0))$ is <u>transitive</u>, meaning, given any two pts $a, b \in D_1(0)$, there is an automorphism Φ such that $\Phi(a) = b$.

Why: φ_a, φ_b. Take $\varphi_{-b} \circ \varphi_a$ ✓

Why "Argument" principle?

Know $e^{\log z} = z$ when $\log z$ is an analytic complex log fcn

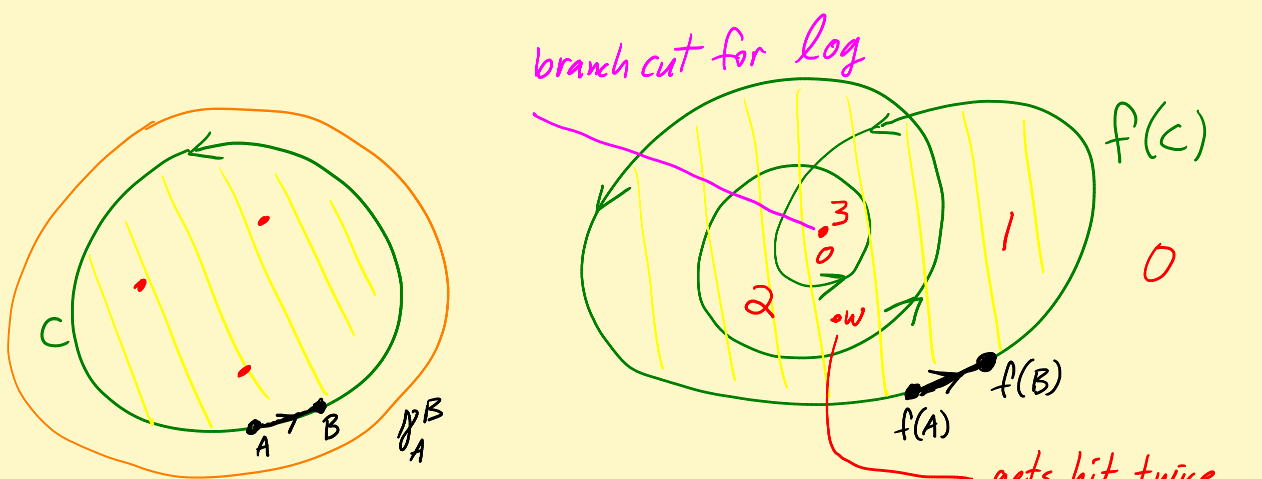
$$\frac{d}{dz}\left(e^{\log z}\right) = 1$$

$$\underbrace{e^{\log z}}_{z}\ \frac{d}{dz}(\log z)$$

<u>Fact</u> $\frac{d}{dz}(\log z) = \frac{1}{z}$, no matter what branch.

Inspiration $\frac{d}{dz}\left[\log f(z)\right] = \frac{1}{f(z)} \cdot f'(z)$ ← Arg princ fcn!

<u>Picture</u>



f analytic inside ◯, non-vanishing on C.

Now $\int_{\gamma_A^B} \underbrace{\frac{f'(z)}{f(z)}}_{\frac{d}{dz}\left[\log f(z)\right]}\, dz = \log f(B) - \log f(A)$

$$= \left[\operatorname{Ln}|f(B)| + i \arg f(B)\right] - \left[\operatorname{Ln}|f(A)| + i \arg f(A)\right]$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right) + i\, \Delta_{\gamma_A^B} \arg f$$

indep of <u>choice</u>

of $\log_\alpha$

Great thing: Add up $\int_{\text{segments}}$ going all the way around.

The $\operatorname{Ln}|f(z_k)|$ terms pairwise cancel, and the first term cancels the last term.

So $$\int_C \frac{f'}{f}\,dz = i\sum \Delta \arg f(z)$$

Divide by $2\pi i$: $$= \underbrace{\frac{1}{2\pi}\sum \Delta \arg f(z)}_{\text{\# times } f(z) \text{ winds around } 0 \text{ as } z \text{ traces out } C \text{ in counterclockwise sense.}}$$

Remark Apply to $F(z) = f(z) - w$: Get

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz = \text{\# times } f(C) \text{ wraps around } w$$

$$= \text{Total \# zeroes of } f(z)-w \text{ inside } C\text{, counted with multiplicity.}$$

Important calculation: $\gamma : z(t)\ ,\ a \le t \le b$

$f(\gamma) : f(z(t))\ ,\ a \le t \le b$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))}{f(z(t))-w}\,z'(t)\,dt$$

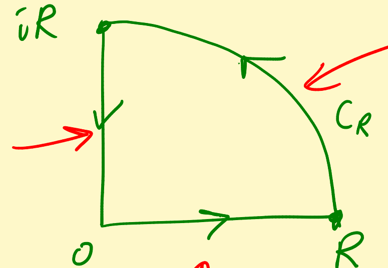
$$= \frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt}\left[f(z(t))\right]}{f(z(t))-w}\,dt$$

$$= \underbrace{\frac{1}{2\pi i}\int_{f(\gamma)} \frac{1}{\zeta - w}\,d\zeta}$$

$= \mathrm{Ind}_{f(\gamma)}(w)$ = winding # of $f(\gamma)$ about w

= # times $f(\gamma)$ winds around w in couterclockwise sense.

<u>Qual prob</u> How many zeroes of $z^{2001} + 1999z + 1$ fall in the first quadrant?



interesting. Take lim of <u>imag part</u>

$$\int_{C_R} \frac{2001z^{2000}+1999}{z^{2001}+1999z+1}\,dz$$

as $R \to \infty$.

Δarg easy

Δarg easiest $= 0$