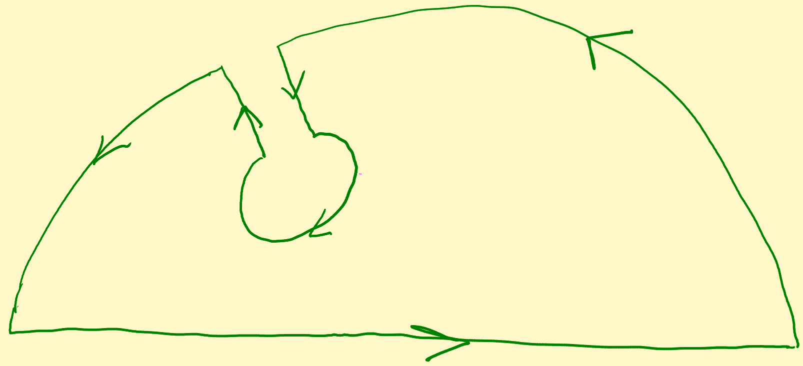
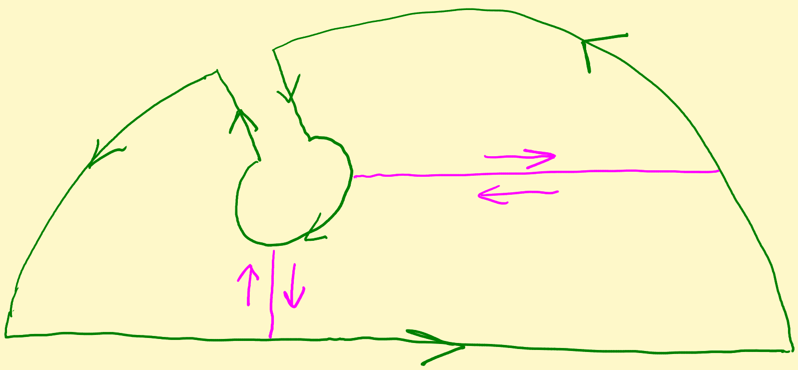
Lecture 21 Fun with the Baby residue theorem on "toy regions"

HWK 5 due Thurs 11 pm

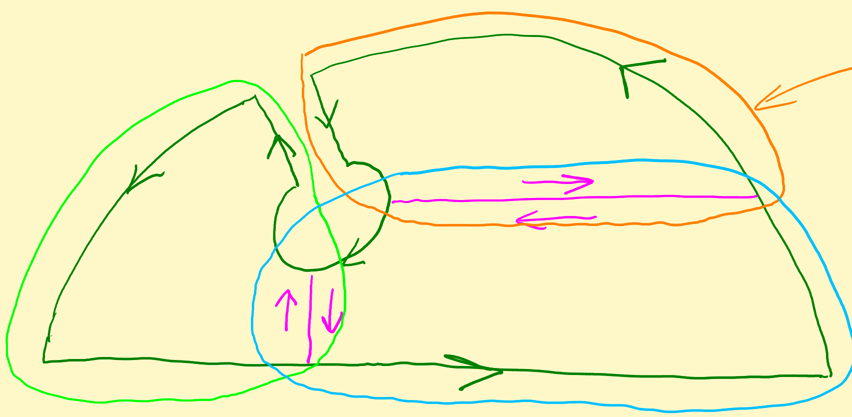


γ simple closed curve consisting of lines and circles



Can see Cauchy thm.

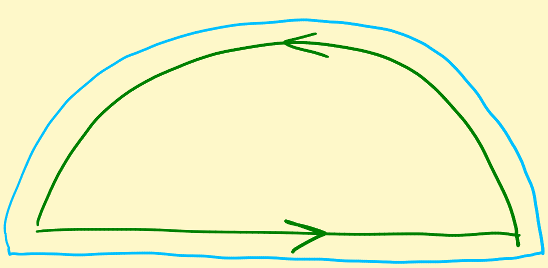
$$\int_\gamma = \sum_1^3 \int_{\Gamma_j} = 0$$

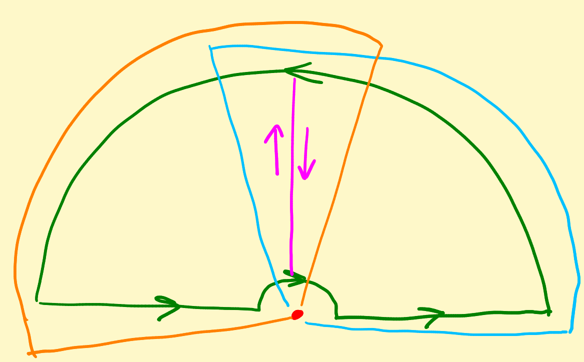


convex open sets

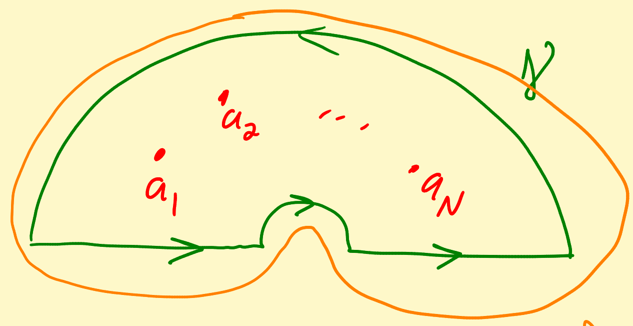
See, if f analytic on open set containing γ and its "inside", then $\int_\gamma f\,dz = 0$.

EX





Baby residue theorem on toys



f analytic on a bigger open set, except at finitely many pts inside γ where f has poles.

Then $$\int_\gamma f\, dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{a_j} f$$

Pf Let $$R_j(z) = \frac{A_N}{(z-a_j)^N} + \cdots + \frac{A_1}{z-a_j} \leftarrow \operatorname{Res}_{a_j} f$$

be the principal part of f at a_j.

Now $f(z) - \sum_{j=1}^{N} R_j(z)$ has removable singularities at a_j's. By giving it proper values at a_j and applying the Cauchy thm, get

$$0 = \int_\gamma f\, dz - \sum_{j=1}^{N} \underbrace{\int_\gamma R_j\, dz}_{= 2\pi i \operatorname{Res}_{a_j} f}$$

Why :

$$R_j(z) = \frac{A_N}{(z-a_j)^N} + \cdots + \frac{A_1}{z-a_j}$$

all others are derivatives.

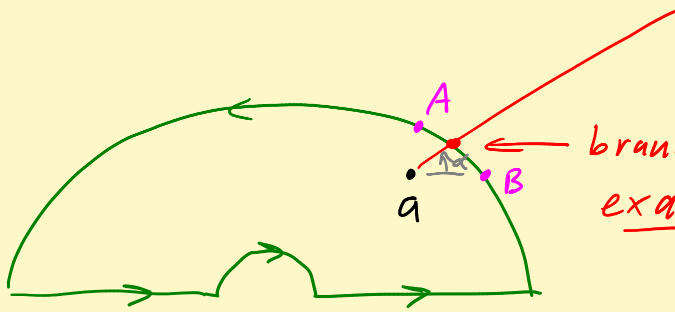
not a derivative

they $\int$ to zero
by Fund Thm Calc.

$$(z-a_j)^k = \frac{d}{dz}\left[\frac{(z-a_j)^{k+1}}{k+1}\right] \quad k \neq -1$$

$$\int_\gamma R_j \, dz = \int_\gamma \frac{A_1}{z-a_j} \, dz = A_1 \cdot 2\pi i$$

Why



branch cut crosses γ in exactly one pt

$$H(a) = \int_\gamma \frac{1}{z-a} \, dz = \lim_{\substack{A \text{ slides down} \\ B \text{ slides up}}} \int_{\gamma_A^B} \frac{1}{z-a} \, dz$$

γ_A^B ← starts at A, follows γ, ends at B.

$$= \lim \int_{\gamma_A^B} \frac{d}{dz}\left[\log_\alpha (z-a)\right] dz$$

$$= \lim \left[\log_\alpha (B-a) - \log_\alpha (A-a)\right]$$

$$= \lim \Big(\ln|B-a| + i\,(2\pi + \alpha - \varepsilon_B)\Big) - \Big(\ln|A-a| + i\,(\alpha + \varepsilon_A)\Big)$$

$$= i\left[(2\pi - \alpha) + \alpha\right] = 2\pi i \quad \checkmark$$

Recall $\displaystyle \frac{d}{da} H(a) = \int_\gamma \frac{d}{da}\left[\frac{1}{z-a}\right] dz = \int_\gamma \frac{1}{(z-a)^2} \, dz$

$$= \int_\gamma \frac{d}{dz}\left[\frac{-1}{z-a}\right] dz = 0$$

$H' \equiv 0 \Rightarrow H(a) = \text{const inside } \gamma = 2\pi i$

Thm Suppose P and Q polynomials and

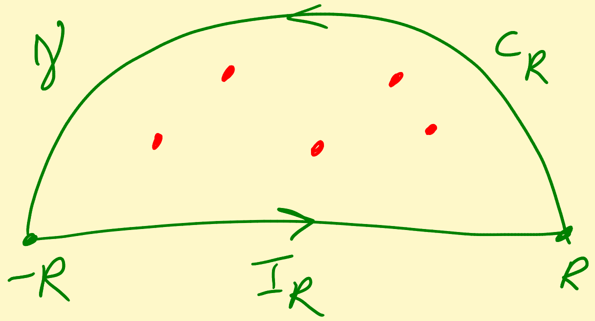
1) $\deg Q \geq \deg P + 2$

2) Q has no zeroes on $\mathbb{R}$.

Then $$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P(z)}{Q(z)}$$

where $Z_Q^+ = \{\text{zeroes of } Q \text{ in } \underbrace{\text{UHP}}_{\text{upper half plane}}\}$

Pf



R big enough that all zeroes of Q in Z_Q^+ are inside γ. $R > R_0$

I_R : $z(t) = t$, $z'(t) = 1$. $-R \leq t \leq R$

$$\int_{I_R} \frac{P}{Q}\, dz = \int_{-R}^{R} \frac{P(t)}{Q(t)} \cdot 1\, dt \longleftarrow \text{want}$$

C_R : $z(t) = Re^{it}$, $0 \leq t \leq \pi$

Claim $\displaystyle\int_{C_R} \frac{P}{Q}\,dz \to 0$ as $R \to \infty$.

Key $\quad a|z|^{\deg P} \le |P(z)| \le A|z|^{\deg P}$, $|z| > R_P$

$\quad b|z|^{\deg Q} \le |Q(z)| \le B|z|^{\deg Q}$, $|z| > R_Q$

$$\left|\int_{C_R} \frac{P}{Q}\,dz\right| \le \left(\max_{C_R} \left|\frac{P}{Q}\right|\right)\cdot(\pi R)$$

$$\le \underbrace{\frac{A R^{\deg P}}{b\, R^{\deg Q}} \cdot \pi R}_{\frac{A\pi}{b}\,\frac{1}{R^{\underbrace{\deg Q - \deg P - 1}_{\ge 1}}}} < \frac{A\pi}{b}\cdot\frac{1}{R}$$

when $R > 1$

when $R > \mathrm{Max}(R_0, R_P, R_Q, 1)$.

So $\left|\int_{C_R}\right| \to 0$ as $R \to \infty$.

Res thm: $\displaystyle\int_\gamma = \int_{I_R} + \int_{C_R} = 2\pi i \sum \mathrm{Res}$

Let $R \to \infty$ $\qquad I + 0 = \checkmark$

Note: $\left|\frac{P(x)}{Q(x)}\right| \leq \frac{C}{|x^2|}$ for $|x| > R > 1$

shows $\int_{-\infty}^{\infty}$ is absolutey integrable. So

$$\int_{-R}^{R} \to \int_{\mathbb{R}} \text{ as } R \to \infty.$$

Computing residues EX Find $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

in UHP

Step 1 Find 4-th roots of -1 : $e^{i\pi/4}, e^{i3\pi/4}$

roots r_1, r_2, r_3, r_4 $e^{i5\pi/4}, e^{i7\pi/4}$

in LHP

$$I = 2\pi i\left[\mathrm{Res}_{e^{i\pi/4}} f + \mathrm{Res}_{e^{i3\pi/4}} f\right] \qquad f(z) = \frac{1}{z^4+1}$$

Hmmm. Near r_1, $f(z) = \frac{1}{(z-r_1)(z-r_2)(z-r_3)(z-r_4)}$

$$= \frac{1}{z-r_1}\left[\underbrace{\frac{1}{\prod_{k\neq 1}(z-r_k)}}_{H(z),\text{ Analytic near } r_1}\right]$$

$= A_0 + A_1(z-r_1) + \cdots$

$$= \frac{A_0}{z-r_1} + \text{regular power series in } (z-r_1)$$

$$A_0 = \operatorname{Res}_{r_1} f = H(r_1) = \frac{1}{(r_1-r_2)(r_1-r_3)(r_1-r_4)}$$ Ugh!

Better idea: Suppose $f(z)$ and $g(z)$ are analytic near a (on a disc about a) <u>and</u> g has a <u>simple</u> zero at a, meaning $g(a)=0$, but $g'(a) \neq 0$.

Then $$\boxed{\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}}.$$

<u>EX</u>: $$\frac{1}{z^4+1}$$

$\leftarrow f(z) \equiv 1$

$\leftarrow g(z) = z^4+1 \leftarrow$

$g'(z) = 4z^3$

$g(r_1) = 0$ ✓

$g'(r_1) = 4r_1^3 \neq 0$

So $$\operatorname{Res}_{r_1} \frac{1}{z^4+1} = \frac{1}{4r_1^3}$$ Not ugh!