Lecture 22 Applications of the residue thm

Thm Suppose f and g are analytic on $D_\varepsilon(a)$ and g has a simple zero at a, meaning that $g(a)=0$ and $g'(a)\neq 0$.

Then $\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$.

Pf:

$$g(z) = \underset{\substack{\uparrow \\ a_0 = g(a) = 0}}{a_0} + \underset{\substack{\uparrow \\ a_1 = \frac{g'(a)}{1!}}}{a_1}(z-a) + a_2(z-a)^2 + \cdots = (z-a)\underbrace{\left[a_1 + a_2(z-a) + \cdots\right]}_{G(z)}$$

[See $G(a) = a_1 = g'(a)$]

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)G(z)} = \frac{1}{z-a}\left[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-a)+\cdots}\right]$$

$$= \frac{\overset{\operatorname{Res}_a}{A_0}}{z-a} + \underbrace{A_1 + A_2(z-a) + \cdots}_{\text{regular power series}}$$

So $\operatorname{Res}_a \frac{f}{g} = A_0 = \frac{f(a)}{G(a)} = \frac{f(a)}{g'(a)}$ ✓

Problem Find a formula for $\operatorname{Res}_a \frac{f}{g}$ when g has a double zero at a, meaning $g(a)=g'(a)=0$, but $g''(a)\neq 0$.

Back to

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left(\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \operatorname{Res}_{e^{i3\pi/4}} \frac{\overset{f(z)\equiv 1}{1}}{\underset{g(z)=z^4+1}{z^4+1}}\right)$$

$$= 2\pi i\left[\frac{1}{4(e^{i\pi/4})^3} + \frac{1}{4(e^{i3\pi/4})^3}\right]$$

$$= \frac{\pi}{2} i\left(\underset{\substack{\uparrow \\ -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}}{e^{-i3\pi/4}} + \underset{\substack{\uparrow \\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}}{e^{-i9\pi/4}}\right) = \frac{\pi}{\sqrt{2}}$$

Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

EX $\quad f(x) = \dfrac{1}{x^2+1}$

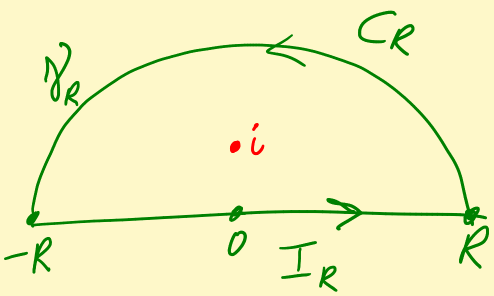
Key: $e^{isz} = e^{is(x+iy)} = e^{-sy}\underbrace{e^{isx}}_{\text{modulus}=1}$

So $\left|e^{isz}\right| = e^{-sy} \begin{cases} <1 & s>0,\ y>0 \\ <1 & s<0,\ y<0 \end{cases}$

Case $s=0$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \operatorname{Tan}^{-1} x \Big|_{-R}^{R} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Case $s>0$ $\quad |e^{isz}| \le 1$ on UHP.



$$f(z) = \frac{e^{isz}}{z^2+1}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \left(\max_{C_R} \frac{|e^{isz}|}{|z^2+1|}\right)(\pi R)$$

Note $|e^{isz}| \le 1$ on C_R

$$|z^2+1| \ge \underbrace{\big||z|^2-1\big|}_{=R^2-1 \text{ when } |z|=R>1}$$

$\leftarrow$ or use $\underline{a|z|^2 \le |z^2+1| \le A|z|^2}$ for $R>R_0$

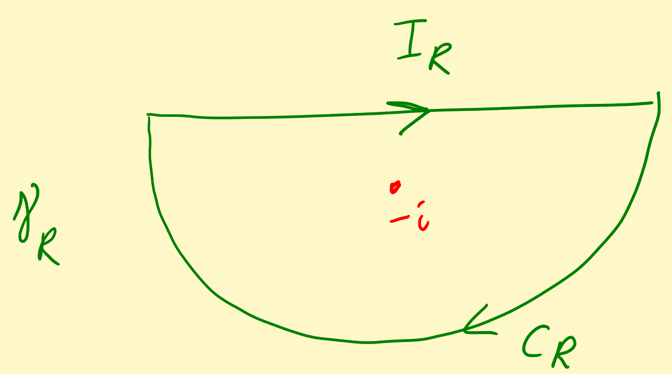
So $\left|\dfrac{e^{isz}}{z^2+1}\right| \le \dfrac{1}{R^2-1}$ on C_R when $R>1$.

Conclude $\left|\int_{C_R}\right| \le \left(\dfrac{1}{R^2-1}\right)(\pi R) \longrightarrow 0$ as $R\to\infty$.

$$\int_{\gamma_R} = \int_{I_R} + \int_{C_R} = 2\pi i \,\mathrm{Res}_i \frac{e^{isz}}{z^2+1} \qquad \begin{array}{l} \leftarrow f(z) \\ \leftarrow g(z) \quad g'(z) = 2z \end{array}$$

Let $R \to \infty$: $\quad \hat{f}(s) + 0 = 2\pi i \dfrac{e^{is(i)}}{2(i)} = \pi e^{-s}$

<u>Case $s < 0$</u> $\quad e^{isz}$ is <u>not</u> bounded on UHP. In fact, it's terrible! But now it is $|e^{isz}| \le 1$ on LHP.

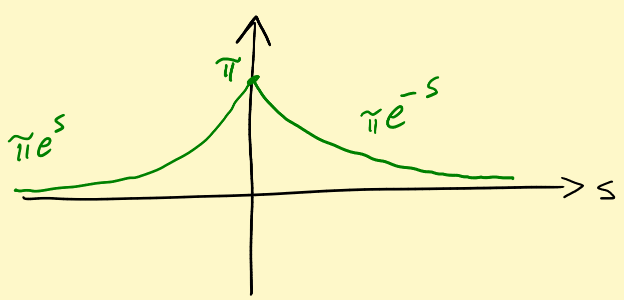


clockwise!

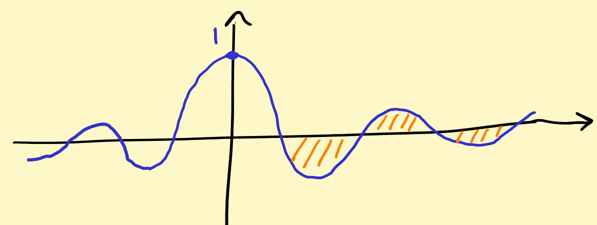
Clockwise Res thm: $\quad \displaystyle\int_{\gamma_R} f(z)\,dz = -2\pi i \,\mathrm{Res}_{-i} \frac{e^{isz}}{z^2+1}$

$$\hat{f}(s) = -2\pi i \frac{e^{is(-i)}}{2(-i)} = \pi e^{s}$$

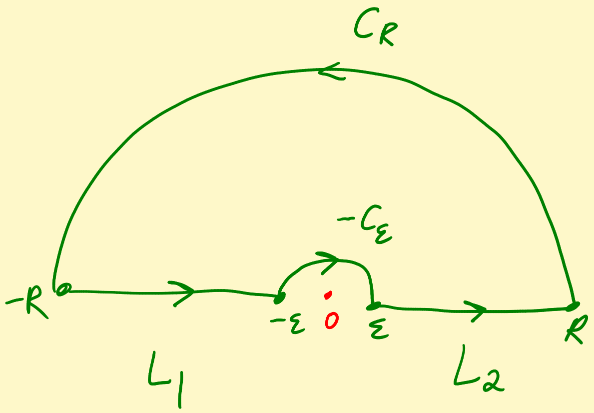
$\hat{f}(s)$



<u>EX</u> $\quad \displaystyle\int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt = \pi$



$\int$ conditionally convergent: Alt series test but not absolutely convergent: $\sum$ positive areas $=\infty$



First try: $f(z)=\frac{\sin z}{z}$.

Ouch! $\sin z$ is terrible on UHP.

Second try: $f(z)=\frac{e^{iz}}{z}$

$$\left(\int_{L_1}+\int_{L_2}\right)\frac{e^{iz}}{z}\,dz=\left(\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}\right)\overbrace{\underset{\substack{\uparrow\\ \text{odd}}}{\frac{\cos t}{t}}+i\underset{\substack{\uparrow\\ \text{even}}}{\frac{\sin t}{t}}}^{\frac{e^{it}}{t}}\,dt$$

$$=2i\int_{\varepsilon}^{R}\frac{\sin t}{t}\,dt \longleftarrow \text{want!}$$

Claim $\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$.

Basic estimate fails us: $\left|\int_{C_R}\right| \le \underbrace{\left(\max_{C_R}\left|\frac{e^{iz}}{z}\right|\right)}_{=\frac{1}{R}}(\pi R)$

happens at $z=\pm R$ on C_R

$\le \pi \not\to 0$. Darn!

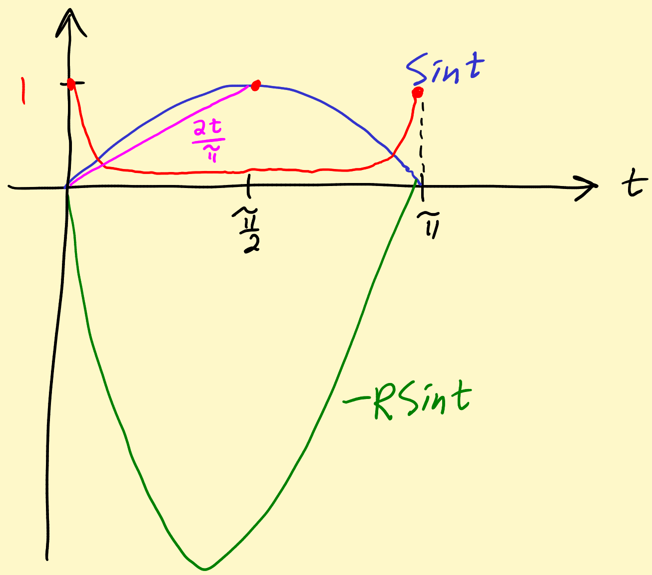
$C_R: z(t)=Re^{it},\ 0\le t\le\pi$

$$\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right|=\left|\int_0^{\pi}\frac{e^{iRe^{it}}}{Re^{it}}\cdot iRe^{it}\,dt\right| \underset{\substack{\uparrow\\ \text{Basic}\\ \int\text{ est}\\ \#1}}{\le} \int_0^{\pi}\left|\quad\right|dt$$

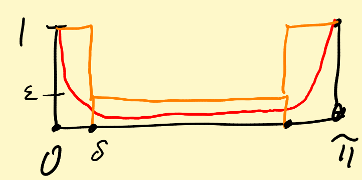
$$\le \underbrace{\int_0^{\pi} \frac{\left|e^{i(R\cos t + iR\sin t)}\right|}{\underbrace{\left|Re^{it}\right|}_{R}}\, R \underbrace{\left|e^{it}\right|}_{1}\, dt}_{\int_0^{\pi} \underbrace{\left|e^{iR\cos t}\right|}_{=1} \underbrace{\left|e^{-R\sin t}\right|}_{e^{-R\sin t}}\, dt}$$

$$\left|\int_{C_R}\right| \le \int_0^{\pi} e^{-R\sin t}\, dt \longrightarrow 0 \quad \text{as } R \to \infty$$

via "Jordan's lemma"



Baby method



Area orange boxs $> \int_0^{\pi} e^{-R\sin t}\, dt$

$$0 \le \frac{2t}{\pi} \le \sin t \quad \text{on } \left[0, \tfrac{\pi}{2}\right]$$

So $$-R\sin t \le -\frac{2Rt}{\pi} \le 0 \quad \text{on } \left[0, \tfrac{\pi}{2}\right]$$

and $$e^{-R\sin t} \le e^{-2Rt/\pi} \le 1 \quad \text{on } \left[0, \tfrac{\pi}{2}\right]$$

symmetry about $x = \frac{\pi}{2}$

Now $$\int_0^{\pi} e^{-R\sin t}\, dt = 2\int_0^{\pi/2} e^{-R\sin t}\, dt$$

$$\le 2\int_0^{\pi/2} e^{-Rt/\pi}\, dt$$

$$= 2\left[-\frac{\pi}{2R}e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right) < \frac{\pi}{R} \quad \text{Yay!}$$

$$\to 0 \text{ as } R \to \infty.$$

Next time: Use <u>simple</u> pole of $\frac{e^{iz}}{z}$ at $z=0$ to

show $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz \to -\pi i$