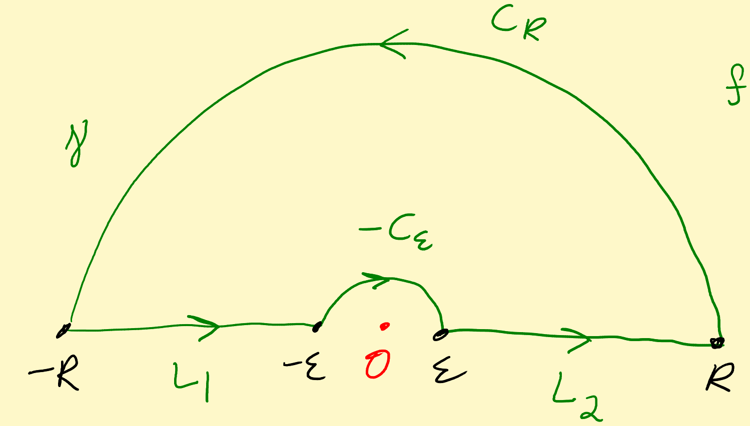
Lecture 23 More residue thm examples



$$f(z) = \frac{e^{iz}}{z}$$

$$I = \int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \pi$$

Last time:

a) $\left(\int_{L_1} + \int_{L_2} \right) f(z)\,dz = 2i \underbrace{\int_\varepsilon^R \frac{\sin t}{t}\,dt}_{\text{want!}}$

b) $\int_{C_R} f(z) \to 0$ as $R \to \infty$

c) Cauchy thm on toy regions: $\int_\gamma f(z)\,dz = 0$

d) Today: Last piece $\int_{-C_\varepsilon} f(z)\,dz \xrightarrow[\varepsilon \to 0]{} \frac{1}{2}(-2\pi i)\underbrace{\operatorname{Res}_0 f(z)}_{=1} = -\pi i$

Finally, near $z=0$ $f(z) = \dfrac{e^{iz}}{z} = \dfrac{\left(1 + iz - \frac{z^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right)}{z}$

$$= \overset{\operatorname{Res}_0 f}{\frac{1}{z}} + \underbrace{i - \frac{1}{2!}z - \frac{iz^2}{3!} + \cdots}_{H(z),\ \text{entire.}}$$

$|H(z)| \le M$ on $\overline{D_1(0)}$.

$$\int_{-C_\varepsilon} f\,dz = -\int_{C_\varepsilon} f\,dz = \underbrace{-\int_{C_\varepsilon} \frac{1}{z}\,dz}_{-\int_0^\pi \frac{1}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt \,=\, -i\pi} - \underbrace{\int_{C_\varepsilon} H(z)\,dz}_{\mathcal{E}}$$

and $|\mathcal{E}| \le \left(\max_{C_\varepsilon} |H|\right)(\pi\varepsilon) \le \pi M \varepsilon$ when $0 < \varepsilon < 1$.

$\longrightarrow 0$ as $\varepsilon \to 0$.

$$\left(\int_{L_1} + \int_{L_2}\right) + \int_{C_R} + \int_{-C_\varepsilon} = 0$$

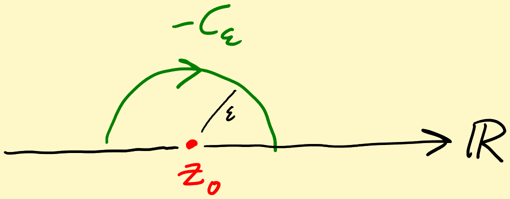
Let $R \to \infty$.

$$\left(2i\int_\varepsilon^\infty \frac{\sin t}{t}\,dt\right) + 0 + \int_{-C_\varepsilon} = 0$$

Next, let $\varepsilon \to 0$.

$$2i\int_0^\infty \frac{\sin t}{t}\,dt + 0 + (-\pi i) = 0 \quad \checkmark$$

General fact



$f(z)$ has a simple pole at z_0

Then $$\int_{-C_\varepsilon} f(z)\,dz \xrightarrow[\varepsilon\to 0]{} \frac{1}{2}(-2\pi i)\cdot \mathrm{Res}_{z_0} f.$$

remember: half the whole circle

Application of Jordan's lemma:

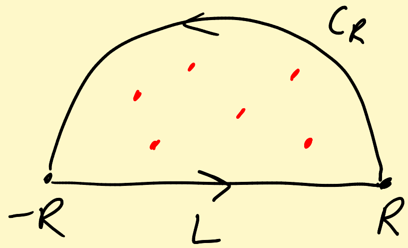
Thm P, Q polys 1) $\deg Q \geq \deg P + 1$ instead of 2

2) Q has no zeroes on $\mathbb{R}$.

Then $$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{ix}\,dx = 2\pi i \sum_{a \in Z_Q^+} \mathrm{Res}_a \frac{P(z)}{Q(z)} e^{iz}$$

zeroes of Q in UHP

Pf



$$\left|\int_{C_R} \frac{P(z)}{Q(z)}\, e^{iz}\, dz\right| \le \underbrace{\frac{A\, R^{\deg P}}{b\, R^{\deg Q}}}_{\text{basic poly estimate}} \underbrace{\left(R\int_0^{\pi} e^{-R\sin t}\, dt\right)}_{\text{Jordan's lemma}}$$

control by $\int_0^{\pi} e^{-R\sin t}\, dt$ which $\to 0$ as $R \to \infty$.

EX $\quad I = \int_0^{2\pi} \sin^4\theta \; d\theta \qquad C_1: \ z(\theta) = e^{i\theta}, \ 0 \le t \le 2\pi$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{e^{i\theta}-\frac{1}{e^{i\theta}}}{2i}$$

Hmmm $\quad I = \int_0^{2\pi} \left[\frac{z(\theta)-\frac{1}{z(\theta)}}{2i}\right]^4 \underbrace{\frac{i e^{i\theta}}{\underbrace{i e^{i\theta}}_{i z(\theta)}}}_{\text{wish I had } z'(\theta) = ie^{i\theta} \text{ here}} d\theta \qquad z'(\theta)\, d\theta$

$$= \int_{C_1} \underbrace{\left[\frac{z-\frac{1}{z}}{2i}\right]^4 \cdot \frac{1}{iz}}_{\text{pole at } z=0} dz \qquad = 2\pi i \,\mathrm{Res}_0$$

$$\frac{1}{i(2i)^4}\cdot\frac{1}{z}\left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)$$

$$\frac{1}{16i}\left(z^3 - 4z + \frac{6}{z} - \frac{4}{z^3} + \frac{1}{z^5}\right)$$

Res term

So $\quad I = 2\pi i \operatorname{Res}_0 = 2\pi i \left(\frac{1}{16 i} \cdot 6 \right) = \frac{3\pi}{4}$

Method $\quad R(z,w)$ rational fcn

Then $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta =$

$$= \int_{C_1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i} \right) \cdot \frac{1}{iz}\, dz .$$

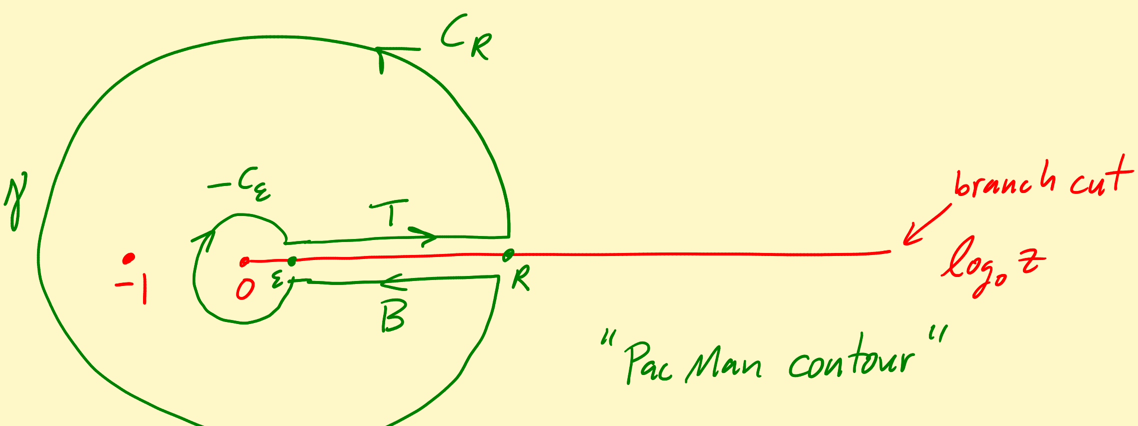
Use residue thm if there are no poles on C_1.

EX $\quad I = \displaystyle\int_0^\infty \frac{x^{-\alpha}}{x+1}\, dx \quad , \quad 0 < \alpha < 1$

$$x^{-\alpha} = e^{-\alpha \ln x} \quad , \quad x > 0 .$$

Hmmm $\quad z^{-\alpha} = e^{-\alpha \log z} \longleftarrow$ branch?

$$f(z) = \frac{e^{-\alpha \log_0 z}}{z+1}$$



"Pac Man contour"

$\log_0 z = \ln|z| + i\theta$
$\theta \in \arg z, \ 0 < \theta < 2\pi$

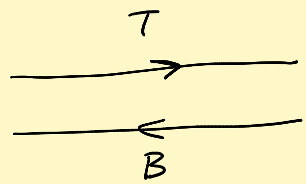
Res thm: $\displaystyle\int_\gamma f\,dz = 2\pi i \operatorname{Res}_{-1} f(z) = 2\pi i \frac{e^{-\alpha \log_0(-1)}}{1 \leftarrow \left.\frac{d}{dz}(z+1)\right|_{(-1)}}$

$z+1$ has a simple zero at $z=-1$.

$$= 2\pi i\, e^{-\alpha(\overset{\operatorname{Ln}|-1|}{0} + i\pi)}$$

$$= 2\pi i\, e^{-i\pi\alpha}$$

Limit as mouth closes:



$\log_0 z = \operatorname{Ln}|z| + i\varepsilon$

$\log_0 z = \operatorname{Ln}|z| + i(2\pi - \varepsilon)$

$$\int_T f(z)\,dz \longrightarrow \int_\varepsilon^R \frac{e^{-\alpha \operatorname{Ln} x}}{x+1}\,dx \leftarrow \text{want!}$$

as T slides down to $\mathbb{R}$

$$\int_B f(z)\,dz = -\int_{-B} f(z)\,dz \to -\int_\varepsilon^R \frac{e^{-\alpha(\operatorname{Ln} x + i2\pi)}}{x+1}\,dx$$

$$= -e^{-i2\pi\alpha} \underbrace{\int_\varepsilon^R \frac{e^{-\alpha \operatorname{Ln} x}}{x+1}\,dx}_{\text{also want!}}$$

Step 1: T down, B up.

Step 2: Let $R \to \infty$ $\leftarrow$ $\displaystyle\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$

Step 3: Let $\varepsilon \to 0$ $\leftarrow$ $\displaystyle\int_{-C_\varepsilon} f(z)\,dz \to 0$ as $\varepsilon \to 0$

Step 4: $\displaystyle\underbrace{\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx}_{I} - e^{-i2\pi\alpha} \underbrace{\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx}_{I} = 2\pi i\, e^{-i\alpha\pi}$

$$I = \frac{2\pi i \; e^{-i\alpha\pi}}{1 - e^{-i2\pi\alpha}} \cdot \frac{e^{i\alpha\pi}}{e^{i\alpha\pi}}$$

$$= \pi \underbrace{\left[\frac{2i}{e^{i\alpha\pi} - e^{-i\alpha\pi}} \right]}_{1/\sin\alpha\pi} = \frac{\pi}{\sin\alpha\pi}$$

Next time : $\int_{C_R} \to 0$, $\int_{C_\varepsilon} \to 0$.

Key : $\left| e^{-\alpha \log_0 z} \right| = \left| e^{-\alpha \operatorname{Ln}|z| - i\alpha\theta} \right|$

$$= \underbrace{\left| e^{-\alpha \operatorname{Ln}|z|} \right|}_{= e^{-\alpha \operatorname{Ln}|z|}} \underbrace{\left| e^{-i\alpha\theta} \right|}_{=1} = |z|^{-\alpha}$$