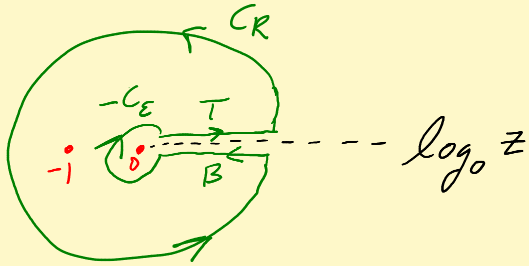
Lecture 24 Linear fractional transformations (LFTs), conformal maps

Last time:

$$\boxed{I = \int_0^\infty \frac{t^{-\alpha}}{t+1}\,dt \qquad 0<\alpha<1}$$



$$f(z) = \frac{e^{-\alpha \log_0 z}}{z+1}$$

$$\left| e^{-\alpha \log z} \right| = \left| e^{-\alpha(\ln|z| + i\theta)} \right| = \underbrace{\left| e^{-\alpha \ln|z|} \right|}_{|z|^{-\alpha}} \cdot \underbrace{\left| e^{-i\alpha\theta} \right|}_{1} = |z|^{-\alpha}$$

$$\left| \frac{e^{-\alpha \log_0 z}}{z+1} \right| \le \frac{|z|^{-\alpha}}{\big||z|-1\big|}$$

$$|z+1| \ge \big||z|-1\big| = \begin{cases} R-1 & \text{if } |z|=R>1 \\ 1-\varepsilon & \text{if } |z|=\varepsilon<1 \end{cases}$$

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{R^{-\alpha}}{R-1}\cdot(2\pi R) \qquad \text{if } R>1$$

$\longrightarrow 0$ as $R\to\infty$ because $\alpha>0$.

$$\left| \int_{-C_\varepsilon} f(z)\,dz \right| \le \frac{\varepsilon^{-\alpha}}{1-\varepsilon}\cdot(2\pi\varepsilon) \qquad \text{if } 0<\varepsilon<1$$

$\longrightarrow 0$ as $\varepsilon\to 0$ because $\alpha<1$.

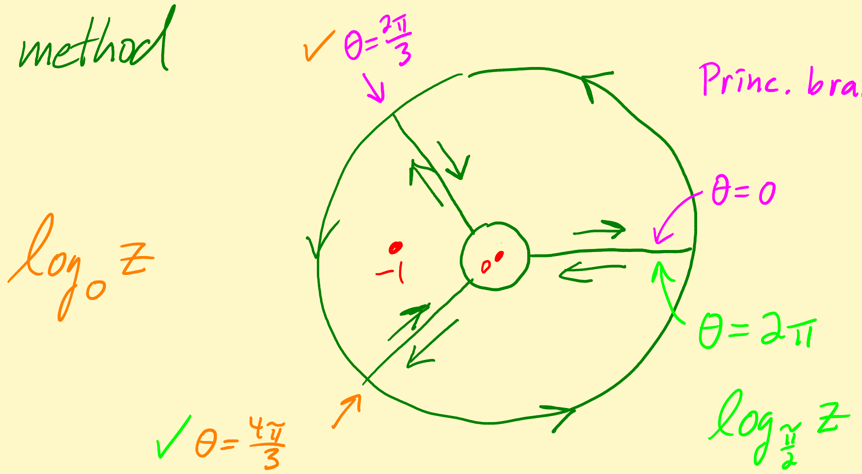
Last steps: Close mouth, let $R\to\infty$, let $\varepsilon \searrow 0$.

$$\underbrace{\left(\int_T + \int_B\right) f(z)\,dz}_{\downarrow \atop (1-e^{-i\alpha 2\pi})\int_{\varepsilon \leftarrow 0}^{R \to \infty} \frac{t^{-\alpha}}{t+1}dt} + \underbrace{\int_{C_R} f\,dz}_{\downarrow \atop 0} + \underbrace{\int_{-C_\varepsilon} f\,dz}_{\downarrow \atop 0} = 2\pi i \cdot \underbrace{e^{-i\alpha\pi}}_{Res_{-1} f(z)}$$

Get $$\int_0^\infty \frac{t^{-\alpha}}{t+1}\,dt = \frac{2\pi i\, e^{-i\alpha\pi}}{1-e^{-i\alpha 2\pi}} \cdot \frac{e^{i\alpha\pi}}{e^{i\alpha\pi}}$$

$$= \pi \frac{2i}{e^{i\alpha\pi} - e^{-i\alpha\pi}} = \underline{\frac{\pi}{\sin \alpha\pi}}$$

Alternate method



Get same answer.

Linear fractional transformations (LFTs) $\qquad L(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

Note $ad-bc=0$: rows of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ lin dependent.

So, either denom $\equiv 0$, or num $= c$(denom). $L(z) \equiv$ const.

Facts $\quad L: \hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$ $\qquad$ Think: LFTs $= Aut(\hat{\mathbb{C}})$

Case $c=0$ $\quad L(z)=\dfrac{az+b}{d} = Az+B$

Then $L(\infty)=\infty$ $\leftarrow$ meaning $\lim_{z\to\infty} L(z)=\infty$

Setting these values makes L cont $\hat{\mathbb{C}}$ ⟲

1-1, onto : $\mathbb{C}$ ⟲ and $\hat{\mathbb{C}}$ ⟲

Case $c\neq 0$ $\quad \lim_{z\to\infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}$ $\quad$ Set $L(\infty)=\dfrac{a}{c}$

$L(z)$ blows up (has a simple pole) when $cz+d=0$

$z=\dfrac{-d}{c}$

Set $L\left(-\dfrac{d}{c}\right)=\infty$

Need to show $L: \mathbb{C}-\left\{\frac{-d}{c}\right\} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C}-\left\{\frac{a}{c}\right\}$

Easy! $\quad w=\dfrac{az+b}{cz+d}$ $\quad$ Solve for $\boxed{z=\dfrac{dw-b}{-cw+a}=L^{-1}(w)}$

Hmmm: $\quad L \sim \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbb{A}}$ $\qquad L^{-1} \sim \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{looks like } \mathbb{A}^{-1}\ldots}$

Aha! $\quad L^{-1} \sim \underbrace{\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\mathbb{A}^{-1}}$ (because mult const in num and denom cancel in LFT)

Fact $\quad L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2$ $\leftarrow$ matrix mult! $\qquad L^{-1} \sim \mathbb{A}^{-1}$

$$z = Iz = \frac{1\cdot z + 0}{0\cdot z + 1} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}$$

Association is a group homomorphism into grp of 2x2 complex matrices with $\det \neq 0$.

Fact LFTs : $\left\{ \begin{matrix} \text{lines} \\ \text{circles} \end{matrix} \right\}$

Lemma LFTs are generated (via composition) by 4 basic maps :

1) $z \mapsto rz, \quad r \in \mathbb{R}^+$ (stretch)

2) $z \mapsto e^{i\theta} z$ (rotation)

3) $z \mapsto z + b, \quad b \in \mathbb{C}$ (translation)

4) $z \mapsto 1/z$ (inversion)

Pf Case $c=0$ $L(z) = \underset{re^{i\theta}}{\underset{\uparrow}{A}} z + B$ 1,2,3 enough

lines $\to$ lines

circles $\to$ circles

Case $c \neq 0$

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\left(b - \frac{ad}{c}\right) \leftarrow Re^{i\varphi}}{\underset{re^{i\theta}}{\underset{\uparrow}{cz}} + d}$$

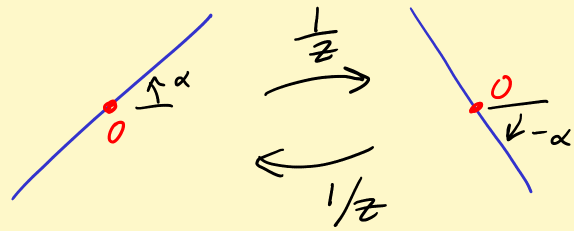
1. stretch
2. rotate
3. translate
4. invert
5. stretch
6. rotate
7. translate

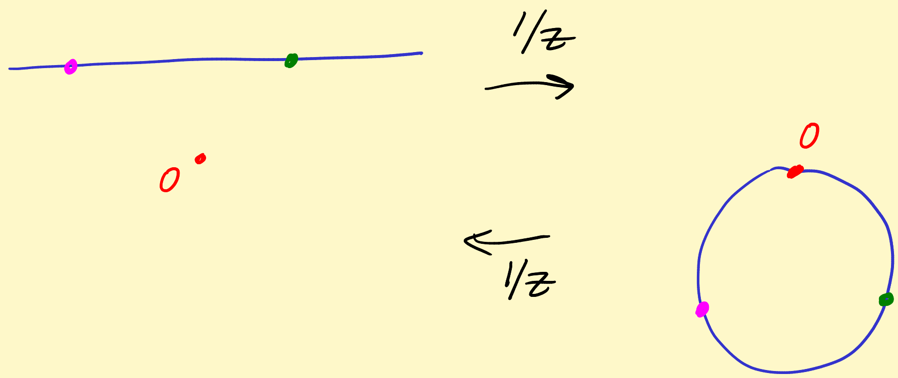
Need 7 maps from 1–4.

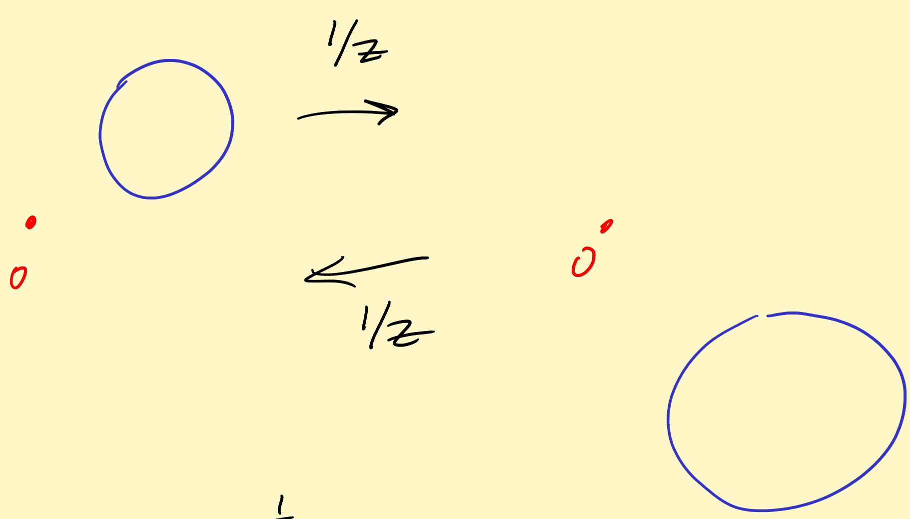
<u>Cor</u> LFTs : $\left\{ \begin{matrix} \text{lines} \\ \text{circles} \end{matrix} \right\}$

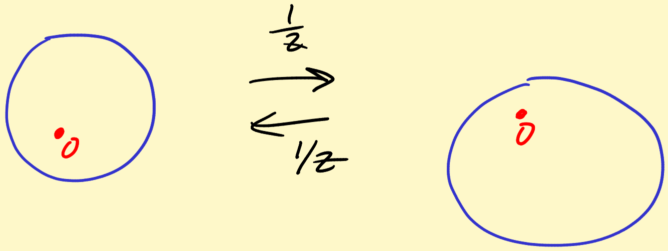
<u>Pf</u> 1, 2, 3 : {lines}   {circles}

4 can mix them up



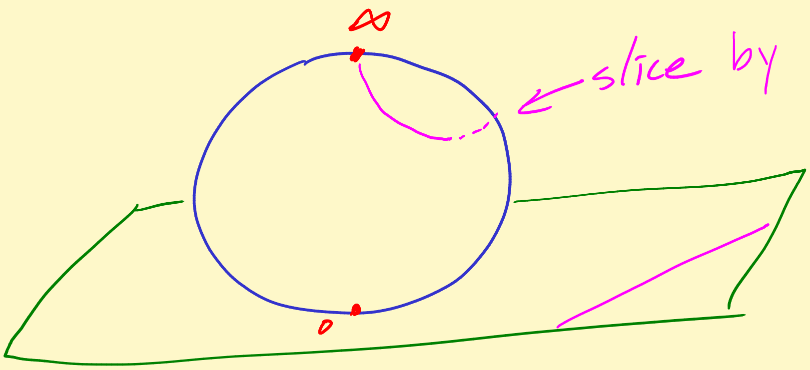






Pf HS analytic geom

Fun fact On the Riemann sphere $\left\{\begin{matrix}\text{lines}\\ \text{circles}\end{matrix}\right\} \sim \{\text{slices}\}$



Lemma A LFT that fixes 3 distinct pts must be the identity.

Pf $L(z_j) = z_j \quad j = 1, 2, 3$

Case $c \neq 0$ $\quad \dfrac{az_j + b}{cz_j + d} = z_j$

$$c z_j^2 + (d-a) z_j - b = 0$$ Quad eqn with $\underline{\underline{3}}$ roots!

$\underbrace{c = 0}$, $d - a = 0$, $b = 0$

det $\neq 0$ so $d \neq 0$. $\quad a = d \quad L(z) = \dfrac{az + 0}{0 \cdot z + a} = z$ ✓

Case $c = 0$, easier. 2 pts enough.

Cor If two LFTs agree at 3 distinct pts, they are the same.

Lucky fact: 3 pts determine a <u>circle</u>.

2 pts in $\mathbb{C}$ plus the pt at ∞ determine a <u>line</u>.

<u>Useful LFT</u>

$$\frac{z-a}{z-b} \quad \begin{cases} a \mapsto 0 \\ b \mapsto \infty \end{cases}$$

$$\left(\frac{c-b}{c-a}\right) \cdot \frac{z-a}{z-b} \quad \begin{cases} a \mapsto 0 \\ b \mapsto \infty \\ c \mapsto 1 \end{cases}$$