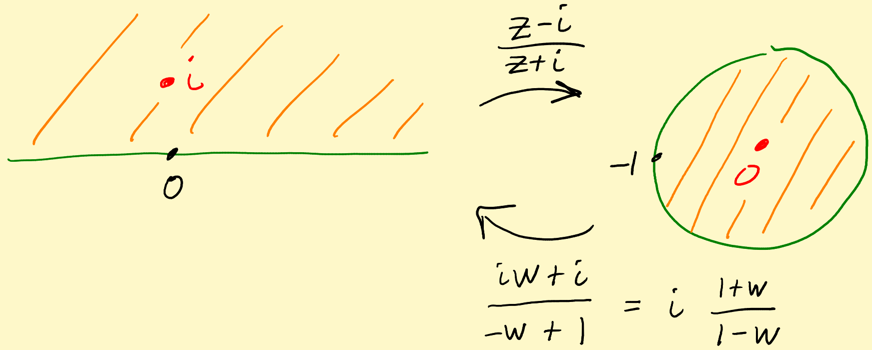
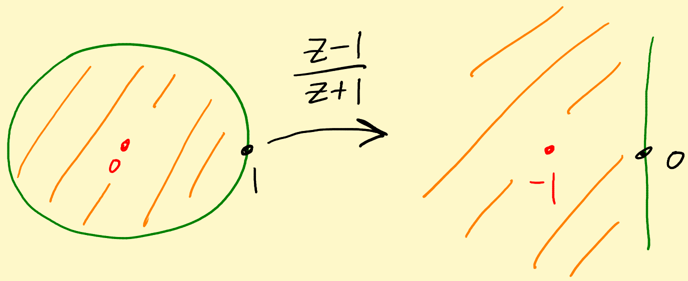
Lecture 26 Schwarz reflection

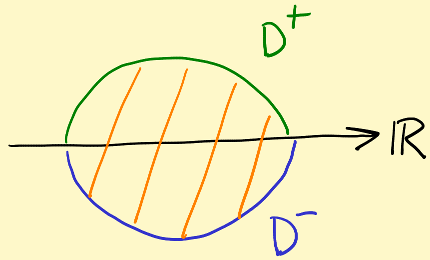
HWK 7 due Thurs, Nov 7

My favorite LFTs :





Lemma



$f : D \to \mathbb{C}$ continuous

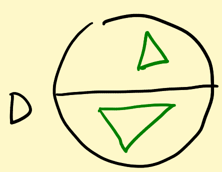
analytic in $D^+ = D \cap \{z : \operatorname{Im} z > 0\}$

analytic in $D^- = \quad \text{. . . .} < 0 \text{. . .}$

Then f is analytic on D

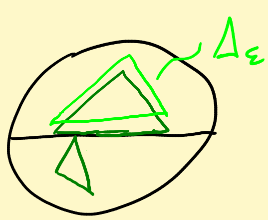
Pf Morera's! $\qquad \Delta \subset D$

Easy cases



$\int_\Delta f \, dz = 0$ by Cauchy's thm.

Next cases



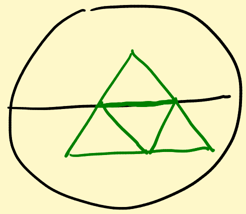
$\Delta \; : \; z(t)$

$\Delta_\varepsilon : \; z_\varepsilon(t) = z(t) \pm i\varepsilon$

Uniform continuity on a bigger compact disc $\bar{\mathcal{D}}$ with $\Delta \subset \mathcal{D} \subset D$.

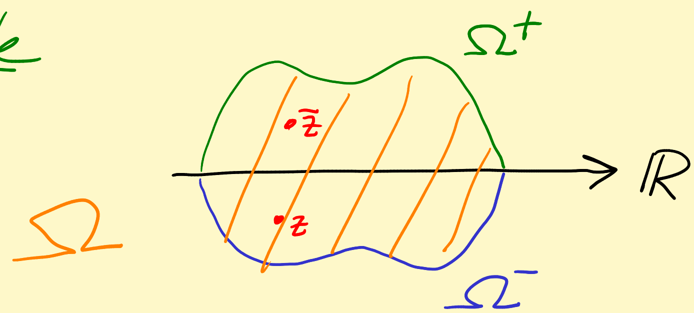
$$\int_\Delta f\,dz = \lim_{\varepsilon\to 0} \underbrace{\int_{\Delta_\varepsilon} f\,dz}_{=0} = 0$$

Last case



$$\int_\Delta = \sum_1^4 \underbrace{\int_{\Delta_j} f\,dz}_{=0} = 0$$

Schwarz reflection principle



Assume f is analytic in $\Omega^+ = \Omega \cap \{z : \operatorname{Im} z > 0\}$

and continuous up to $\mathbb{R}$, and real valued

along $\mathbb{R}$. Then

$$F(z) = \begin{cases} f(z) & \text{in } \Omega^+ \\ f(x) & x \in \mathbb{R}\cap\Omega \\ \underbrace{\overline{f(\bar z)}}_{\text{reflected fcn}} & z \in \Omega^- \end{cases}$$

Pf f cont up to $\mathbb{R}$ and real on $\mathbb{R}$ $\Rightarrow$ F

continuous on Ω. $f(z)$ analytic on Ω^+

$\Rightarrow$ $\overline{f(\bar z)}$ is analytic on Ω^- (HWK 5 : prob 2)

Lemma $\Rightarrow$ F analytic on Ω.

Remark: $f(x+iy) = u(x,y) + iv(x,y)$ on Ω^+

$F(x+iy) = \overline{f(x-iy)} = u(x,-y) - iv(x,-y)$ on Ω^-

Aha! v vanishes on $\mathbb{R}$, so no jump along $\mathbb{R}$.

Remark HWK 5: prob 2: f analytic $\Rightarrow$ $\overline{f(\bar{z})}$ analytic

$$f(x+iy) = u(x,y) + iv(x,y) = \sum_0^\infty a_n (z-z_0)^n$$

$$\overline{f(x-iy)} = \underbrace{u(x,-y) - iv(x,-y)}_{\text{C-R equs} \checkmark} = \overline{\sum_0^\infty a_n (\bar{z}-z_0)^n}$$

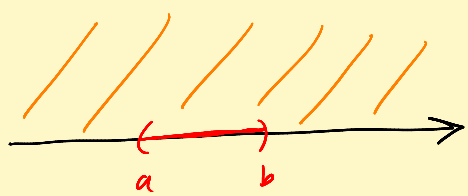
$$= \underbrace{\sum_0^\infty \bar{a}_n (z-\bar{z}_0)^n}_{\text{same RoC by Hadamard}}$$

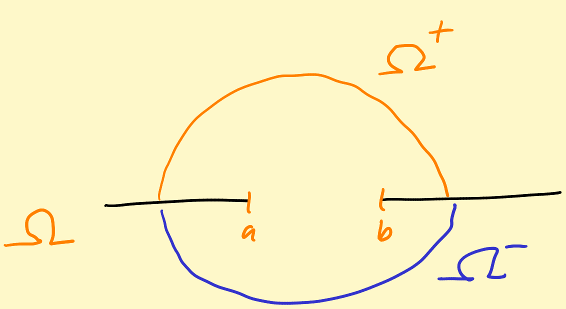
or

Consequence of Schwarz Suppose f is analytic on UHP and continuous up to $(a,b) \subset \mathbb{R}$ and zero there.

Then $f \equiv 0$ on UHP.

Pf

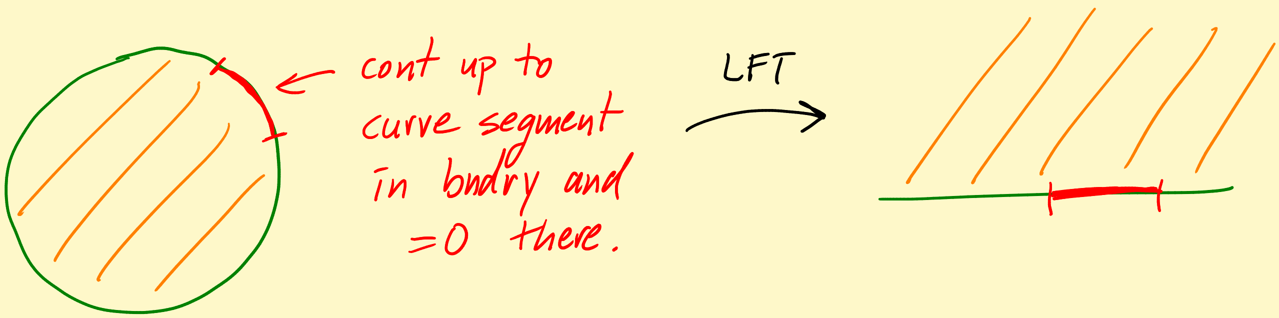




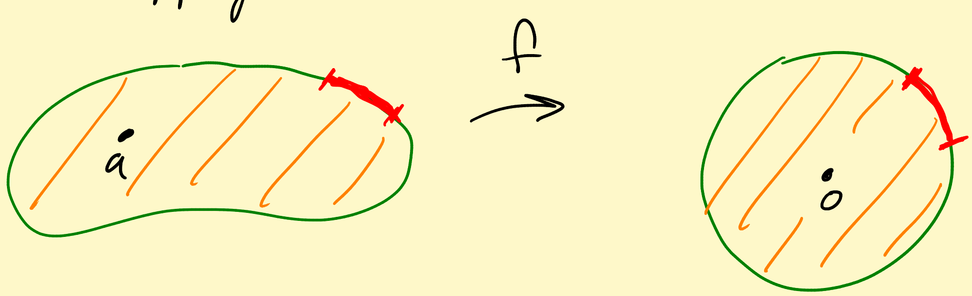
Schwarz $\Rightarrow$ get analytic fcn F on Ω, $F = f$ on Ω^+.

Every pt of (a,b) is a limit pt of Ω, and $F \equiv 0$ on (a,b). Identity thm $\Rightarrow$ $F \equiv 0$ on Ω $\Rightarrow$ $f \equiv 0$ on Ω^+. Identity thm $\Rightarrow$ $f \equiv 0$ on UHP.

Tricks using conformal mapping



Riemann mapping thm



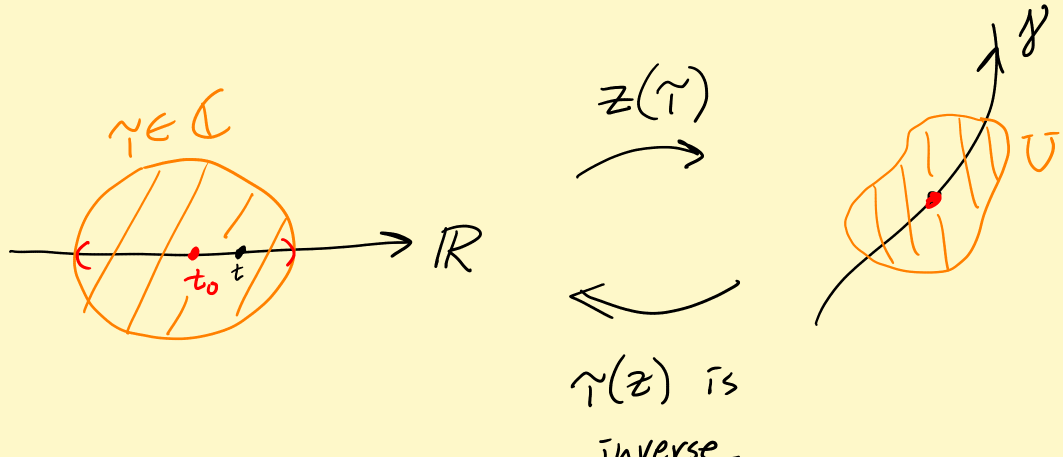
Hmmm. If Ω is bounded by a Jordan curve (simple closed continuous curve), is the Riemann map <u>continuous</u> up to the boundary?

Yes! Carathéodory's thm.

Also f: 1-1 onto Jordan cure $\leftrightarrow$ $C_1(0)$, and f^{-1} cont up $C_1(0)$ too.

Real analytic curves and general Schwarz reflection

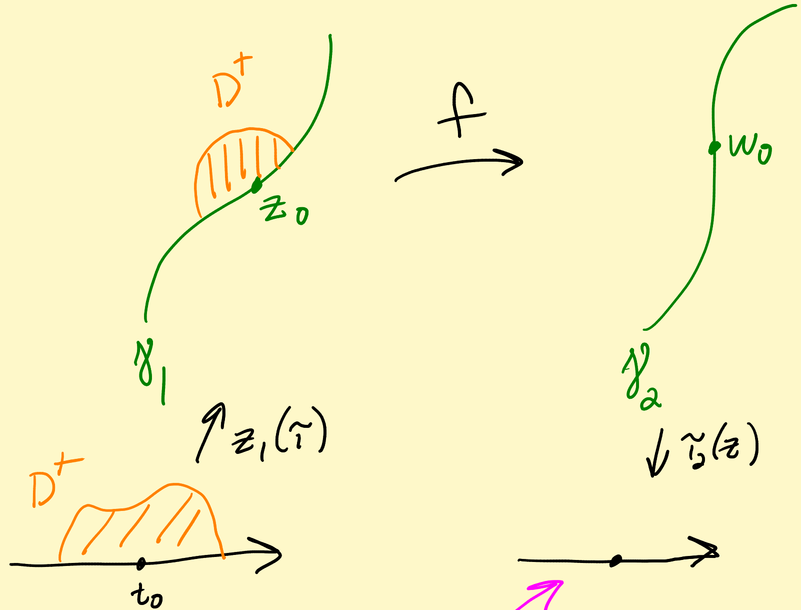
γ real analytic: $z(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^n \qquad c_n \in \mathbb{C}$



Can shrink disc about t_0 to get $z(\tau)$ 1-1 onto analytic map: disc $\to U$

$z'(t_0) \neq 0$.

Cor



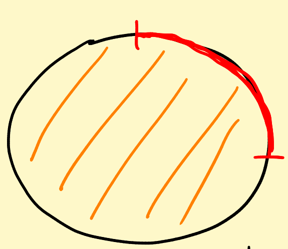
composition is real valued along $\mathbb{R}$.

Use basic Schwarz.

γ_1, γ_2 real analytic curves. f analytic on D^+ side and extends continuously up to γ_1 and $f(\gamma_1) \subset \gamma_2$.

Then f extends analytically to a nbhd of z_0.

Fun exercise



f analytic, continuous up to $C_1(0)$, zero on red.

$F(z) = f(z) f(iz) f(-z) f(-iz) \equiv 0$ on $C_1(0)$.

Max princ $\Rightarrow F \equiv 0$ on $D_1(0)$.

HWK 6 prob $\Rightarrow$ at least one factor $\equiv 0$.

$\Rightarrow f \equiv 0$ on $D_1(0)$