Lecture 28 More about Laurent expansions

HWK 7 due Nov 7

Suppose z_0 is an isolated singularity of f, so f analytic on $D_r(z_0) - \{z_0\}$ for some $r > 0$. $0 < \rho < r$.

$C_\rho(z_0)$

$$a_{-1} = \frac{1}{2\pi i} \int_{C_\rho(z_0)} f \, dz = \mathrm{Res}_{z_0} f$$

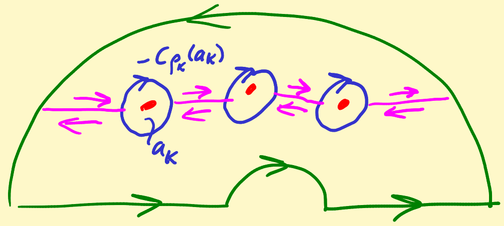
Hmmm

$$\int_{C_\rho(z_0)} f \, dz = 2\pi i \, \mathrm{Res}_{z_0} f$$

A residue thm for fcns with an <u>essential</u> singularity.

<u>Feeling</u> Residue thm on circles $\Rightarrow$ more general residue thm.

γ



f analytic inside, on γ with isolated sing at <u>finitely</u> many pts inside γ

$$\underbrace{\int_{\substack{\text{Top closed}\\ \text{curve}}}}_{\substack{=0\\ \text{Cauchy on Toys}}} + \underbrace{\int_{\substack{\text{Bottom closed}\\ \text{curve}}}}_{=0} = \left(\int_\gamma - \sum_{k=1}^{N} \underbrace{\int_{C_{\rho_k}(a_k)} f \, dz}_{2\pi i \, \mathrm{Res}_{a_k} f} \right)$$

Get
$$\int_\gamma f \, dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}_{a_k} f \, .$$

<u>Meaning of residue</u> <u>Thm</u> Suppose f analytic on $D_R(z_0) - \{z_0\}$.

$\mathrm{Res}_{z_0} f = 0 \iff f$ has an analytic antiderivative on $D_R(z_0) - \{z_0\}$.

Pf ($\Leftarrow$) Easy. If $f = F'$ on $D_R(z_0) - \{z_0\}$, then

$$\operatorname{Res}_{z_0} f = \frac{1}{2\pi i}\int_{C_\rho(z_0)} \underbrace{f}_{=F'}\, dz = 0 \quad \text{by Fund Thm Calc.} \checkmark$$

($\Rightarrow$) Assume $\int_{C_\rho(z_0)} f\, dz = 0$. $\Longleftrightarrow$ $\operatorname{Res}_{z_0} f = 0$.

Lemma If f is analytic on a domain Ω, f has an analytic antiderivative on $\Omega \Longleftrightarrow \int_\gamma f\, dz = 0$ for any closed curve γ in Ω.

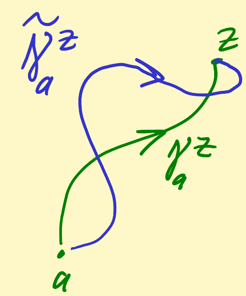
Pf: ($\Rightarrow$) Easy. Fund Thm Calc.

($\Leftarrow$) Hmmm. On convex open set $F = \int_{L_a^z} f\, dw$ is an antiderivative.

Big idea: "Define" $F(z) = \int_{\gamma_a^z} f(w)\, dw$ $\leftarrow$ a fixed in Ω

where γ_a^z is any curve from a to z in Ω.

Step 1 F is well defined because

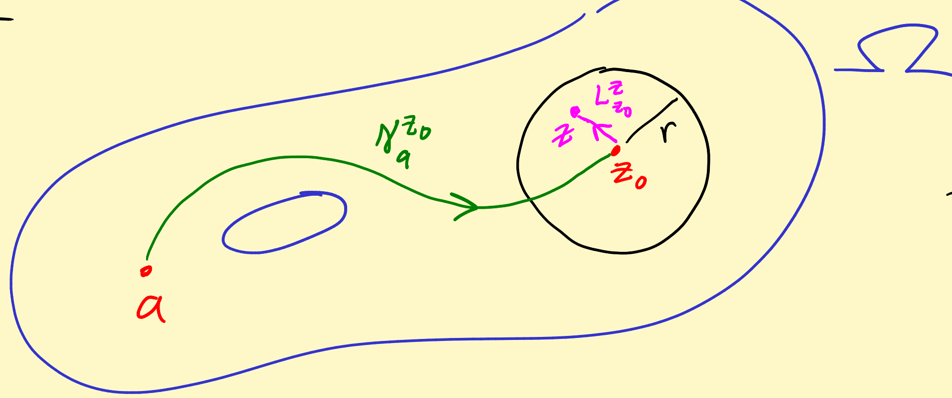


$$F(z) - \tilde{F}(z) = \left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right) f\, dw$$

$$= \underbrace{\left(\int_{\gamma_a^z} + \int_{-\tilde{\gamma}_a^z}\right)}_{\int_\Gamma} f\, dw = 0$$

$\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z) \leftarrow$ closed curve in Ω.

Step 2 Standard trick



$D_r(z_0) \subset \Omega$, Ω open.

$$F(z) = \left(\int_{\gamma_a^{z_0}} + \int_{L_{z_0}^z} \right) f \, dw$$

$$= \underbrace{\left(\int_{\gamma_a^{z_0}} f \, dw \right)}_{\text{const}} + \underbrace{\left(\int_{L_{z_0}^z} f \, dw \right)}_{\text{antiderivative for } f \text{ on convex } D_r(z_0)} \quad \checkmark$$

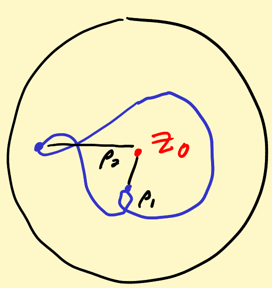
So $F' = f$ on $D_r(z_0)$.

But z_0 was arbitrary. $\checkmark$

Back to Pf of thm. Suppose γ is a closed curve in $D_R(z_0) - \{z_0\}$. Suppose $a_{-1} = \operatorname{Res}_{z_0} f = 0$

$$\int_\gamma f \, dz = \int_\gamma \underbrace{\sum_{n \neq -1} a_n (z - z_0)^n}_{\substack{\sum_0^\infty \text{ and } \sum_{-2}^{-\infty} \text{ converge} \\ \text{uniformly on } \overline{A(\rho_1, \rho_2)}}} dz$$

Laurent expansion



where $\rho_1 = \min \{ |z(t) - z_0| \}$
$\rho_2 = \operatorname{Max} \{ |z(t) - z_0| \}$ $\quad \gamma : z(t)$

$$\underset{\substack{\uparrow\\ \text{unif}\\ \text{conv}}}{=} \sum_{n\neq -1} a_n \underbrace{\int_\gamma (z-z_0)^n \, dz}_{\substack{=0\\ \text{since } n\neq -1.}} = 0. \quad \text{Lemma} \Rightarrow f = F' \checkmark$$

Rouché's thm for a disc. (Works for a "toy region" too.)

Suppose f, g are analytic on a disc $D_R(a)$ and suppose $0 < r < R$ and

$$(*) \quad |f(z) - g(z)| < |f(z)| \quad \text{on} \quad C_r(a).$$

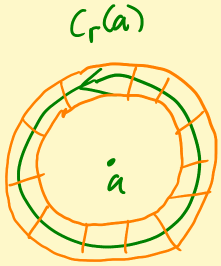
Then f and g have the same number of zeroes inside $C_r(a)$, counted with multiplicity.

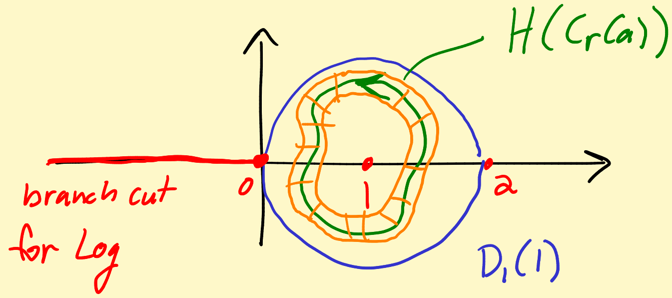
Pf $(*)$ RHS $\Rightarrow$ f has no zeroes on $C_r(a)$.

$(*) \Rightarrow g$ has no zeroes on $C_r(a)$ too.

Step 1 Divide $(*)$ by $|f(z)|$ =

$$\left| 1 - \underbrace{\frac{g(z)}{f(z)}}_{H(z)} \right| < 1 \quad \text{on} \quad C_r(a)$$





$$\frac{1}{2\pi i}\int_{C_r(a)} \frac{H'(z)}{H(z)}\,dz = \underbrace{\frac{1}{2\pi i}\int_{C_r(a)} \frac{d}{dz}\Big(\operatorname{Log} H(z)\Big)\,dz}_{=0 \text{ by Fund Thm Calc}}$$

analytic on ▨ annulus

Fact $\frac{H'}{H} = \frac{f'}{f} - \frac{g'}{g}$ "Logarthmic differentiation"

$$0 = \frac{1}{2\pi i}\int_C \frac{H'}{H}\,dz = \underbrace{\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz}_{\#\text{ zeroes } f} - \underbrace{\frac{1}{2\pi i}\int_C \frac{g'}{g}\,dz}_{\#\text{ zeroes } g}$$

Remark 1) Same pf works for toy regions.

2) Can also assume f and g have finitely many poles inside γ (and none on curve).

Rouché's ineq $\Rightarrow$

$[(\#\text{ zeroes of } f) - (\#\text{ poles of } f)] =$ Same for g.

Fun thing: Rouché's $\Rightarrow$ Fund Thm Alg.

$$f(z) = a_N z^N \qquad g(z) = a_N z^N + \cdots + a_1 z + a_0$$

$$\underbrace{|f(z) - g(z)|}_{\text{poly of deg } \le N-1} \overset{?}{<} \underbrace{|f(z)|}_{\deg N} \quad \text{on } C_R(0)\ ?$$

Basic poly estimate $\Rightarrow$ Rouché's ineq for big enough R.

EX How many zeroes does $z^5 + 3z^4 + \underbrace{\underbrace{8z^3}_{f(z)} + z^2 + z + 1}$ have in the unit disc?

$$\underbrace{z^5 + 3z^4 + 8z^3 + z^2 + z + 1}_{g(z)}$$

Ans 3, counted with mult.

$$|f(z) - g(z)| = |z^5 + 3z^4 + z^2 + z + 1| \le \underbrace{1 + 3 + 1 + 1 + 1}_{7} \quad \text{when } |z| = 1$$

$$< 8 = \underbrace{|8z^3|}_{|f(z)|} \quad \text{when } |z| = 1$$