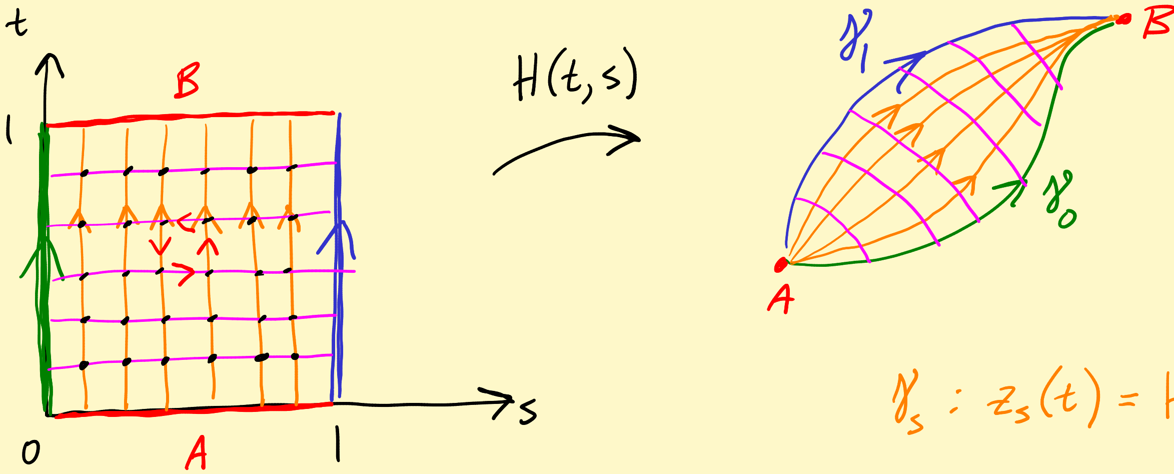
Lecture 30 Homotopy, simply connected domains, The Cauchy thm

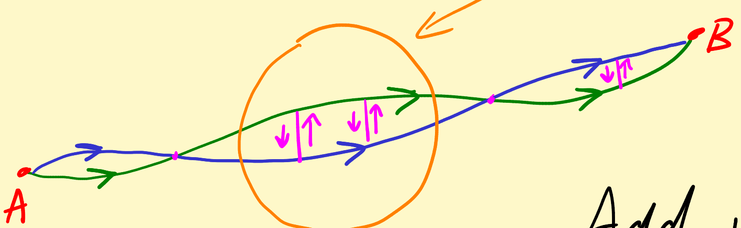


$\gamma_s : z_s(t) = H(t,s)$

Feeling: If γ_0 is "close" to γ_1 in Ω and f is analytic in Ω, the $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$.

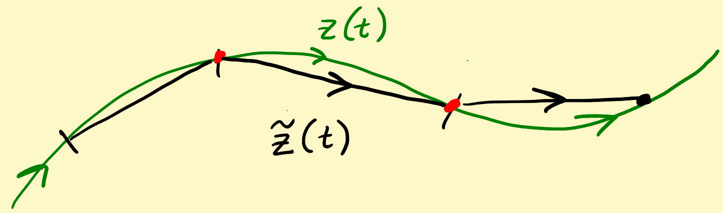
Do Cauchy on convex: Disc $\subset \Omega$

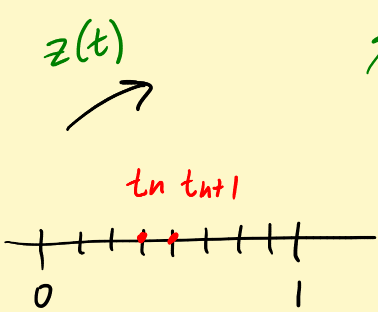
Why:



Add up. Cuts cancel.

Step 1



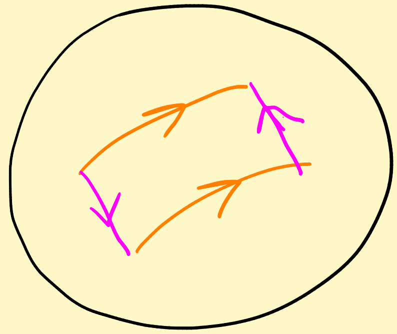


$$\tilde{z}(t) = z(t_n) + \frac{t - t_n}{t_{n+1} - t_n}\left[z(t_{n+1}) - z(t_n)\right]$$

Get piecewise linear approx to a continuous curve.

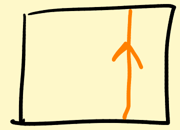
Step 2: Fix grid in our homotopy so that $\exists\, \rho > 0$ such that

(Uniform continuity of H on compact $[0,1]\times[0,1]$.)

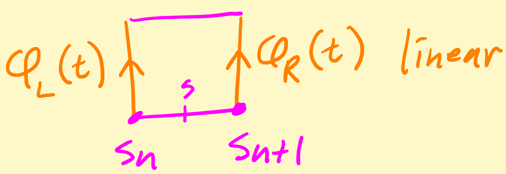


$\overline{D_\rho(a)} \subset \Omega$

Step 3: Do piecewise linear approx on



Step 4: Change def of $H(t,s)$:



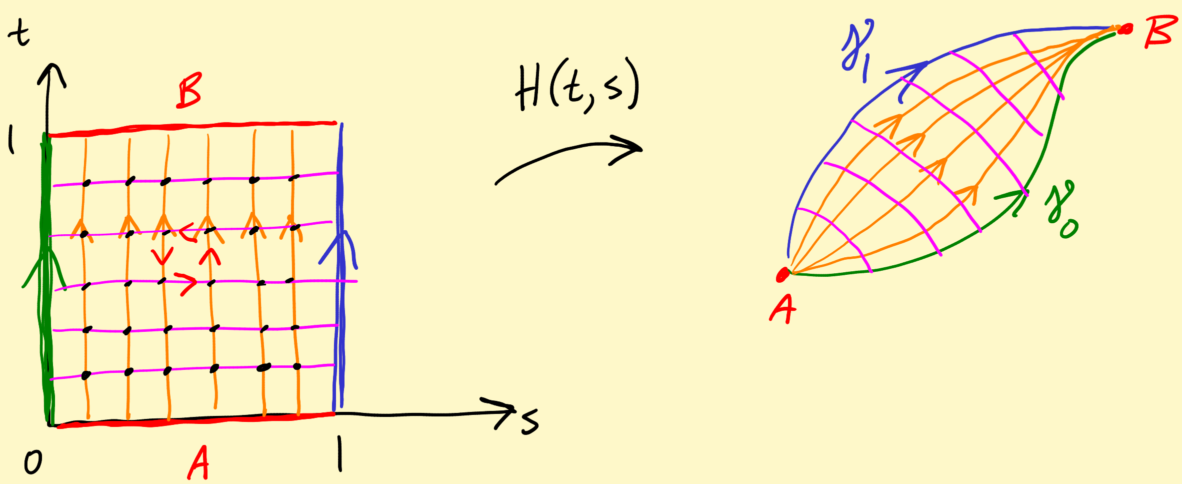
$$\underbrace{\varphi_s(t) = \varphi_L(t) + \frac{s-s_n}{s_{n+1}-s_n}\left[\varphi_R(t) - \varphi_L(t)\right]}_{\text{new def of } H(t,s) \text{ on little square}}$$

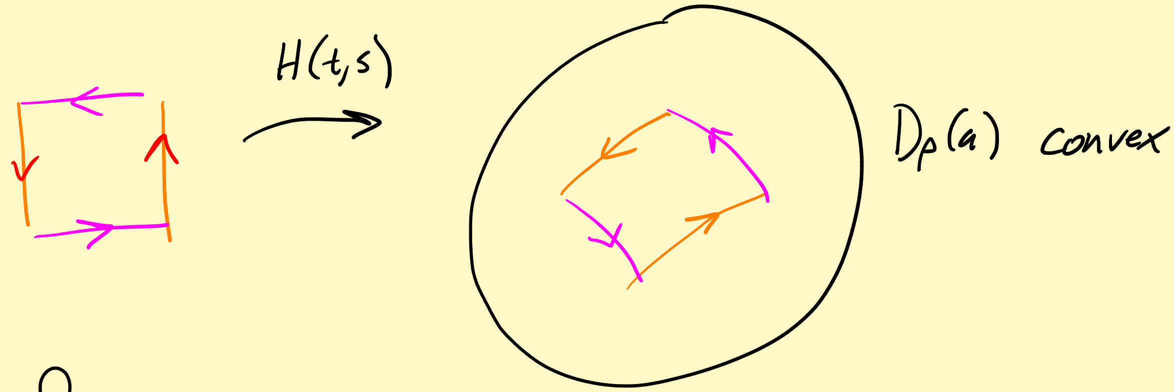
Get new piecewise linear homotopy of $\gamma_0 \sim \gamma_1$

Thm: f analytic on a domain, $\gamma_0 \sim \gamma_1$ in Ω.

Then $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$. ($A, B$ fixed endpoints. $\int$ is "path indep")

Pf





See $\int_{\text{curvelinear rec}} f\,dz = 0$ by Cauchy on convex.

Add up all integrals around squares. Inner sides all cancel!

$$\int_{\text{BOTTOM}} = \int_0^1 f(A)\,\underbrace{\frac{d}{ds}[A]}_{\equiv 0}\,ds = 0$$

$$\int_{\text{RIGHT SIDE}} = \int_0^1 f(\underbrace{H(t,1)}_{z_1(t)})\,\underbrace{\frac{\partial}{\partial t}H(t,1)}_{z_1'(t)}\,dt = \int_{\gamma_1} f\,dz$$

$$\int_{\text{TOP}} = 0 \quad \text{too.}$$

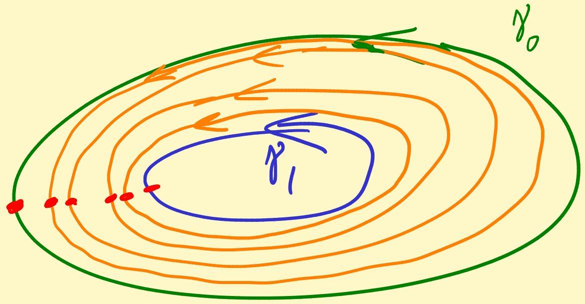
$$\int_{\text{LEFT SIDE}} = -\int_{\gamma_0} f\,dz.$$

Need

Get $\int_{\gamma_0} = \int_{\gamma_1}$ ✓

<u>Def^n</u>: Two <u>closed</u> curves γ_0 and γ_1 in Ω are <u>homotopic</u> in Ω if there is a homotopy $H(t,s)$ such that

$H : [0,1] \times [0,1] \to \Omega$ continuous.



$\gamma_0 : z_0(t), \; 0 \le t \le 1$

$\gamma_1 : z_1(t), \; 0 \le t \le 1$

$\gamma_s : z_s(t) = H(t,s)$

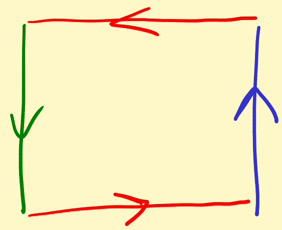
$$\begin{cases} H(t,0) = z_0(t) \\ H(t,1) = z_1(t) \end{cases}$$

$H(0,s) = H(1,s)$ $\leftarrow$ γ_s closed continuous curves.

<u>Thm</u> f analytic on a domain Ω, γ_0, γ_1 $\leftarrow$ piecewise C^1-smooth closed curves

<u>closed</u> homotopic curves, then $\int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz$.

<u>Pf</u>: Same as above but



$$\int_{\text{BOTTOM}} = \int_0^1 f(H(0,s)) \frac{\partial}{\partial s} H(0,s) \, ds$$

because $H(0,s) = H(1,s)$

$$-\int_{\text{TOP}} = \int_0^1 f(H(1,s)) \frac{\partial}{\partial s} H(1,s) \, ds = -\int_{\text{TOP}}$$

Cancel! $\left(\int_{\text{BOTTOM}} + \int_{\text{TOP}} \right) = 0$

<u>Def^n</u>: A domain Ω is simply <u>connected</u>

if every closed curve in Ω is "homotopic to a point" in Ω. $\left[\gamma \sim z_0 : \ \gamma_0 : z(t) \equiv z_0,\ 0 \le t \le 1 \right]$

<u>Fact</u>: $\gamma_0 : z(t) \equiv z_0$, $0 \le t \le 1$

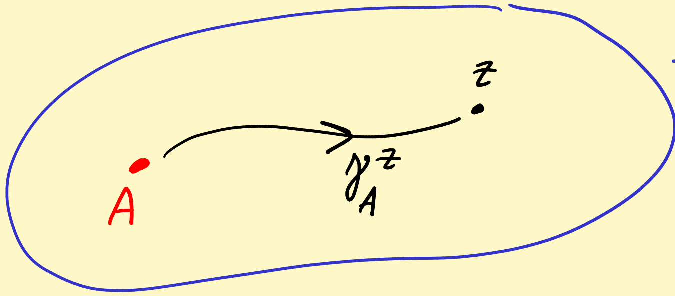
$$\int_{\gamma_0} f\, dz = 0 \text{ for } \underline{\text{any}}\ f.$$

<u>The</u> Cauchy thm: f analytic on a simply connected domain Ω and γ is a closed curve in Ω, then $\int_\gamma f\, dz = 0$.

And if γ_0 and γ_1 are curves in Ω that start at A and go to B, then $\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz$ (because $\gamma_0 \cup (-\gamma_1)$ is a closed curve). "$\int_{\gamma_A^B}$ indep of path"

<u>Theorem</u>: All the theorems we proved on convex open sets now are seen to hold on simply connected domains.

<u>Pf</u>: All those results depended on cooking analytic antiderivatives. Know how to get analytic antiderivatives on simply connected domains:

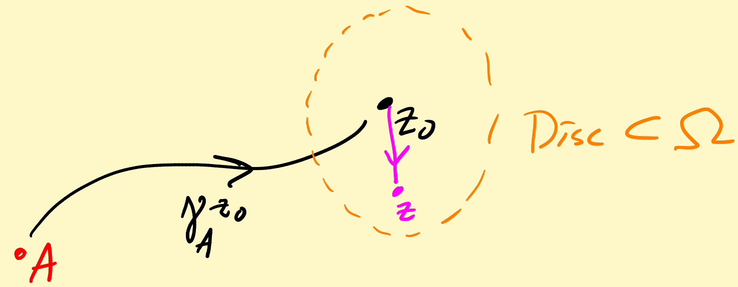


Ω s.c. domain

$$F(z) = \int_{\gamma_A^z} f \, dw \longleftarrow$$

Cauchy thm $\Longrightarrow$ <u>well</u> defined

$\gamma_A^z \cup (-\tilde{\gamma}_A^z)$ closed curve



$$F(z) = \underbrace{\int_{\gamma_A^{z_0}} f \, dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^z} f \, dw}_{\text{antiderive for } f}$$

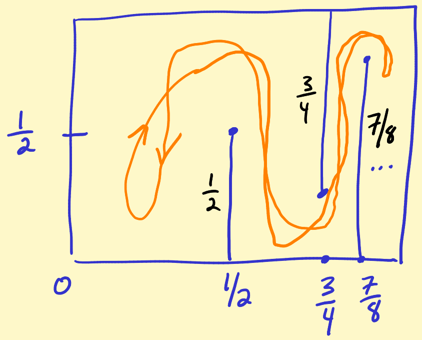
✓

<u>Thm</u> Ω s.c. domain

1) Analytic antiderivatives exist for analytic fcns
2) g non-vanishing analytic. $\exists$ analytic f with $e^f = g$ on Ω, and
3) $\exists$ h analytic with $h^N = g$ on Ω.
4) u harmonic on Ω. $\exists$ harmonic conj v for u <u>on Ω</u>. $\left(\begin{array}{l} f = u_x - i u_y \\ F' = f. \text{ etc.} \end{array} \right)$

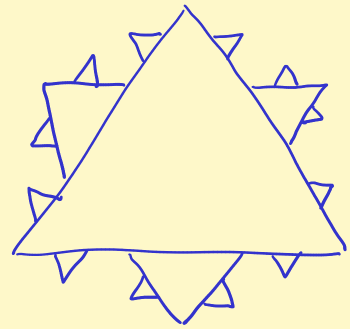
Warning: holes in Ω kill all 4 results.

EX of s.c. domains



"suck it in like spaghetti"

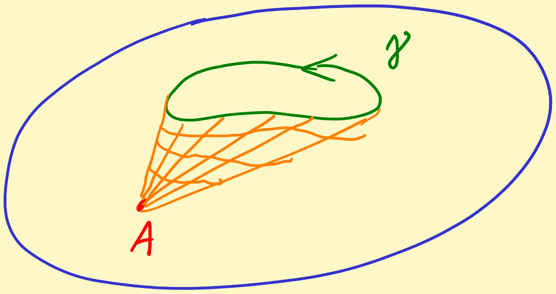
Sierpinsky snowflake



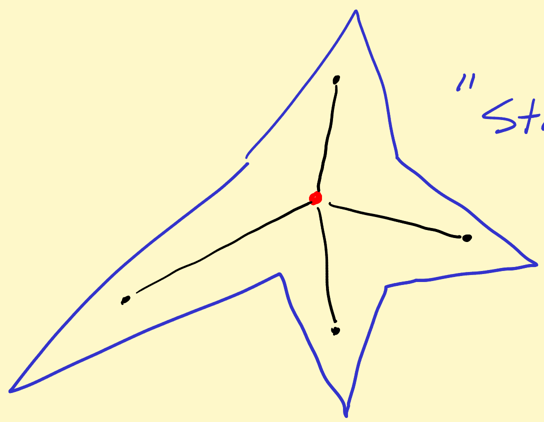
etc.

Isoperimetric ineq $A \leq \underbrace{\frac{L^2}{4\pi}}_{\to \infty}$ as points added.

Convex Ω

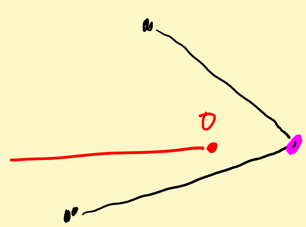


"Tornado homotopy"



"Star shaped"

Same homotopy.



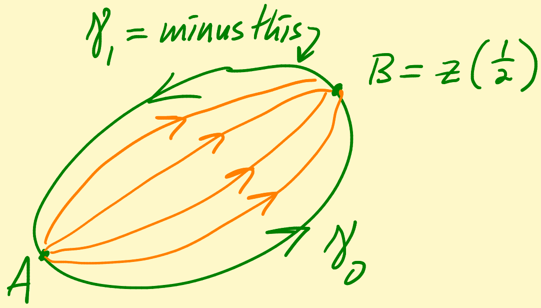
$\mathbb{C} - (-\infty, 0]$ is star shaped.

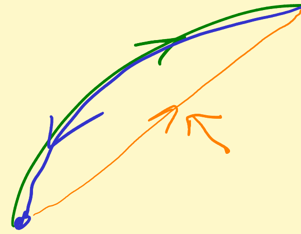
Topology facts 1) Ahlfors: equiv def^n of $\Omega \subset \mathbb{C}$ simply connected: $\hat{\mathbb{C}} - \Omega$ is connected.

Remark: Need $\hat{\mathbb{C}}$ instead of $\mathbb{C}$ when Ω is unbounded, but $\mathbb{C}$ works in place of $\hat{\mathbb{C}}$ when Ω is bounded.

2) Ω is s.c. $\Longleftrightarrow$ given $A, B \in \Omega$ and γ_A^B, $\tilde{\gamma}_A^B$. Then $\gamma_A^B \sim_\Omega \tilde{\gamma}_A^B$.

$(\Leftarrow)$

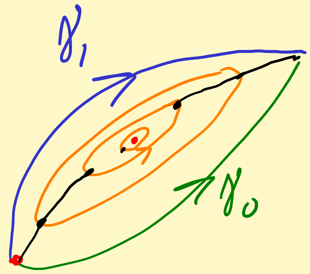




~ out and back.

suck it in like spaghetti

$(\Rightarrow)$



use $-\gamma_1$. Get closed curve

$\gamma_0 \cup (-\gamma_1)$

Can make sense out of $\int_\gamma f \, dz$ when γ is merely continuous.