Lecture 33 The Riemann map

Case $\Omega = \mathbb{C}$. Liouville's $\Rightarrow$ $f: \Omega \to D_1(0)$ analytic $\equiv$ const.

Case $\Omega \neq \mathbb{C}$, simply connected and bounded, then $f(z) = z/M$ is a 1-1 analytic map $\Omega \to D_1(0)$ for big enough const $M > 0$.

Case Ω, s.c., $\neq \mathbb{C}$, unbounded

How to cook up $f: \Omega \to D_1(0)$ analytic and 1-1 (so that $\mathcal{F} \neq \emptyset$): $\Omega \neq \mathbb{C}$, so $\exists\ z_0 \in \mathbb{C} - \Omega$.

So $z - z_0$ is non-vanishing on s.c. Ω. There exists an analytic fcn $g(z)$ on Ω with $g(z)^2 = z - z_0$.

Claim: g is 1-1 on Ω:

$$g(z_1) = g(z_2)$$
$$g(z_1)^2 = g(z_2)^2$$
$$z_1 - z_0 = z_2 - z_0$$
$$z_1 = z_2 \checkmark$$

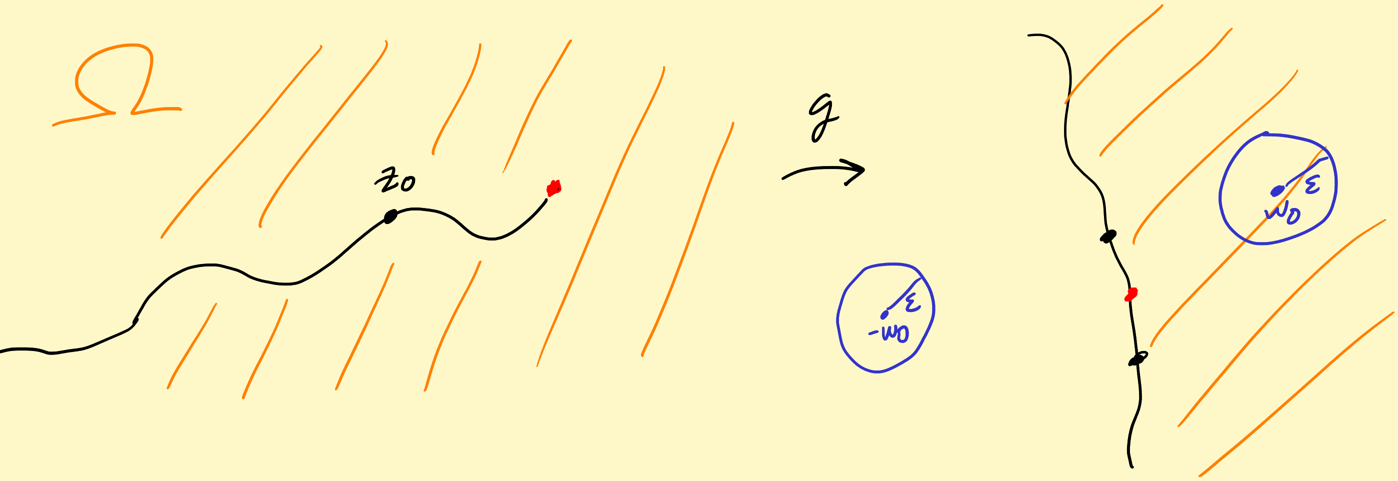
Claim g misses a disc on Ω.

Why: g is not constant. Open Mapping Theorem $\Rightarrow$ $\exists\ D_\varepsilon(w_0) \subset g(\Omega)$.

Sub Claim: $-D_\varepsilon(w_0) \cap g(\Omega) = \emptyset$ [where $-S = \{-z : z \in S\}$]

Because $g(z)^2 = z - z_0$ is 1-1.

If $D_\varepsilon(w_0)$ and $-D_\varepsilon(w_0)$ in $g(\Omega)$, then $g(z)^2$ isn't 1-1. $\lightning$.

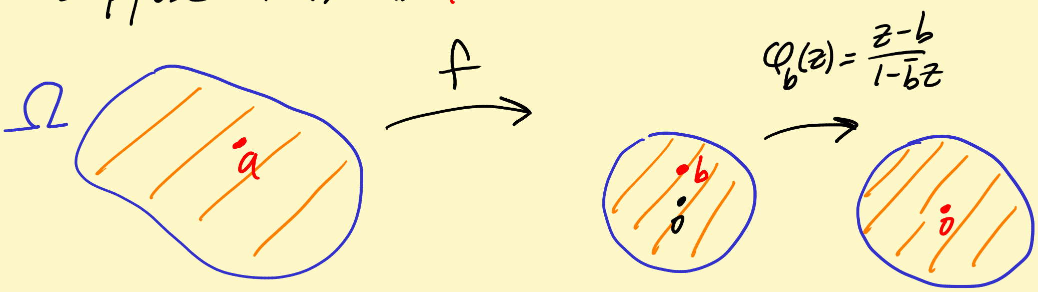


is a 1-1 analytic map into $D_1(0)$.

So $\tilde{\mathcal{F}} \neq \emptyset$.

Fact The mapping $f: \Omega \to D_1(0)$ that we constructed in the proof of the RMT with $|f'(a)| = \text{Max}\{|h'(a)| : h \in \tilde{\mathcal{F}}\}$ is such that $f(a) = 0$.

Why: Suppose $f(a) = b \neq 0$.



Let $F(z) = \varphi_b(f(z))$. Claim $|F'(a)| > |f'(a)|$ ↯

because $F'(a) = \underbrace{\varphi_b'(b)}_{\frac{1}{1-|b|^2} > 1} \cdot f'(a)$

Question How many maps in $\mathcal{F}$ assume Max deriv at a?

Ans f is unique up to rotations.

Schwarz: Suppose f_1, f_2 both Riemann maps for a.

Then $\underbrace{f_1 \circ f_2^{-1}}_{\Phi} : D_1(0) \xrightarrow[\text{onto}]{1\text{-}1} D_1(0)$, fixes $z=0$.

$|\Phi'(0)| \le 1$

$\underbrace{|\Phi^{(-1)\prime}(0)|}_{\frac{1}{|\Phi'(0)|}} \le 1$

So $|\Phi'(0)| = 1$. Schwarz $\Rightarrow$

$\Phi(z) \equiv \lambda z$.

$|\lambda| = 1$.

Conclude: $f_1(z) = e^{i\theta} f_2(z)$.

To get the Riemann map, take a Riemann map with

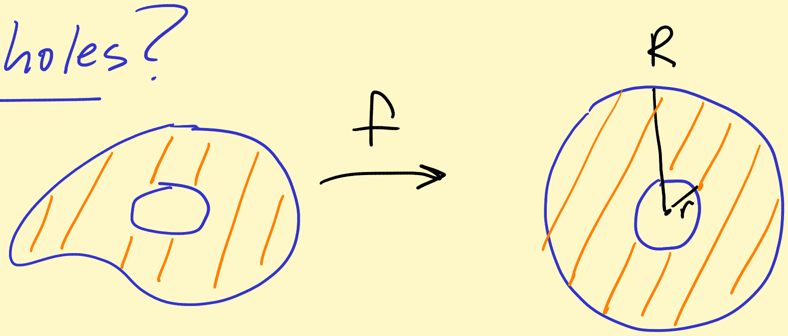
$f'(a) \in \mathbb{R}^+$.

Fact Given $a \in \Omega \neq \mathbb{C}$ s.c., $\exists$ unique analytic

$f : \Omega \xrightarrow[\text{onto}]{1\text{-}1} D_1(0)$ with $f(a) = 0$ and $f'(a) \in \mathbb{R}^+$.

What if Ω has holes?

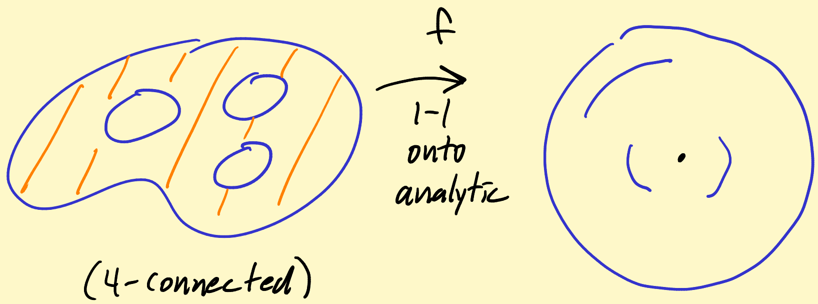
Ω 2-connected:



Fact If two 2-connected domains are conformally equivalent, then $\frac{r_1}{R_1} = \frac{r_2}{R_2}$. ($\frac{r}{R}$ = "modulus of Ω")

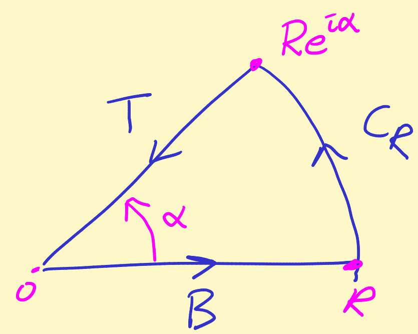
Annulus: Lots of automorphisms: $e^{i\theta} z$, one like $\frac{1}{z}$ called an inversion, but not as big as $\mathrm{Aut}(D_1(0))$. In particular, it is not transitive.

n-connected Ω



$\mathrm{Aut}(\Omega)$ can be just the identity map $f(z)=z$ or finite.

Review problem Find $\displaystyle\int_0^\infty \frac{dt}{1+t^5}$



$\int_B \leftarrow$ want.

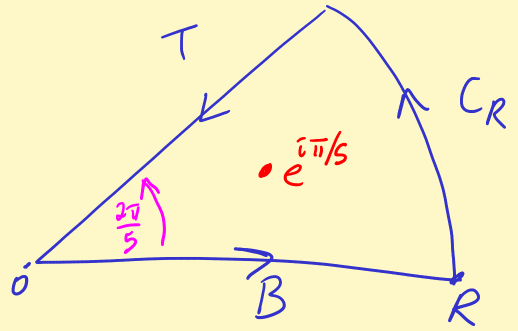
$-T: z(t) = te^{i\alpha}, \; 0 \le t \le R.$
$z'(t) = e^{i\alpha}$

$$\int_T = -\int_{-T} \frac{1}{1+z^5}\,dz = -\int_0^R \frac{1}{1+(te^{i\alpha})^5} e^{i\alpha}\,dt$$

$$= -e^{i\alpha} \int_0^R \frac{1}{1+(e^{i5\alpha})t^5}\,dt$$

Aha! Want $e^{i5\alpha} = 1$. Good choice: $5\alpha = 2\pi$

$\alpha = \frac{2\pi}{5}$



Show $\left| \int_{C_R} \right| \le \frac{1}{R^5 - 1} \cdot \left(\frac{2\pi}{5} R \right)$

if $R > 1$

$\longrightarrow 0$ as $R \to \infty$.

$$\left(1 - e^{i2\pi/5}\right) I + 0 = 2\pi i \operatorname{Res}_{e^{i\pi/5}} \frac{1}{1+z^5} \begin{matrix} \leftarrow f(z) = 1 \\ \leftarrow g(z) \end{matrix}$$

$$= 2\pi i \left. \frac{f(a)}{g'(a)} \right|_{a = e^{i\pi/5}}$$

$$= 2\pi i \cdot \frac{1}{5\left(e^{i\pi/5}\right)^4}$$

$$I = \frac{\pi}{5} \cdot \frac{1}{1 - e^{i2\pi/5}} \cdot \frac{2i}{e^{i4\pi/5}} \cdot \frac{-e^{-i\pi/5}}{-e^{-i\pi/5}}$$

$$= \frac{\pi}{5} \frac{2i}{e^{i\pi/5} - e^{-i\pi/5}} = \frac{\pi/5}{\sin \pi/5}$$