Lecture 34 Harmonic functions, averaging property, Poisson integral formula

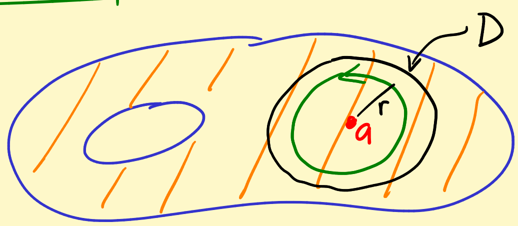
Def^n A real valued fcn u on a domain Ω is harmonic if it is C^2-smooth and $\Delta u \equiv 0$ on Ω.

Fact u is harmonic on Ω $\Longleftrightarrow$ u is locally the real part of an analytic fcn on Ω (meaning for each $z_0 \in \Omega$, there is a $D_\varepsilon(z_0) \subset \Omega$ and h analytic on $D_\varepsilon(z_0)$ such $u = \operatorname{Re} h$ on $D_\varepsilon(z_0)$).

Fact Can get global harmonic conjugates on simply connected Ω.

EX $\ln|z|$ is harmonic on $\Omega = \mathbb{C} - \{0\}$, but has no global harmonic conjugate on Ω. (Holes are bad for this problem.)

Important property harmonic fcns satisfy the averaging property:



$$u(a) = \frac{1}{2\pi r}\int_0^{2\pi} u(a + re^{i\theta})\, \underbrace{r\,d\theta}_{ds}$$

with or without r

Why: Get harm conj for u on D. $f = u + iv$ analytic

Cauchy integral formula for $f(a)$:

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

Take real part to get ave prop for u.

Thm Ave prop $\Rightarrow$ max princ

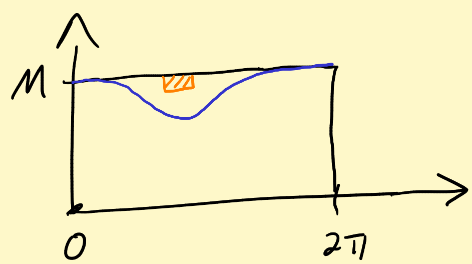
Why Suppose u has a local max at $a \in \Omega$.

So $\exists\, D_\varepsilon(a) \subset \Omega$ such that $u(z) \le u(a) = M$

on $D_\varepsilon(a)$. Ave prop $\underset{M}{\underset{\parallel}{u(a)}} = \frac{1}{2\pi}\int_0^{2\pi} \underset{\le M}{u}(a + \underset{0<r<\varepsilon}{r}e^{i\theta})\,d\theta$

$$\le \frac{1}{2\pi}\int_0^{2\pi} M\,d\theta \le M$$

Calculus lemma



Conclude that $u(a+re^{i\theta}) \equiv M$ for $0 \le \theta \le 2\pi$ for each

r with $0 \le r < \varepsilon$. So $u \equiv M$ on $D_\varepsilon(a)$.

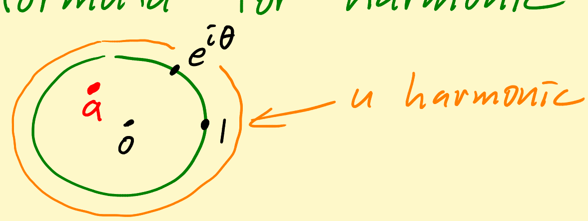
Claim $u \equiv M$ on $D_\varepsilon(a) \Rightarrow u \equiv M$ on Ω.

Pf: $u_x - i u_y$ is analytic on Ω and $\equiv 0$ on $D_\varepsilon(a)$.

Identity thm $\Rightarrow u_x - i u_y \equiv 0$ on Ω, so $\nabla u \equiv 0$

on Ω. $\Rightarrow u$ is const on Ω. $\equiv M$.

Want a "Cauchy integral formula" for harmonic fcns.



$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|a-e^{i\theta}|^2}\, u(e^{i\theta})\, d\theta$$ ← Poisson integral formula

MA 530 home page: "Something about Poisson and Dirichlet" easy and quick way to get this formula.

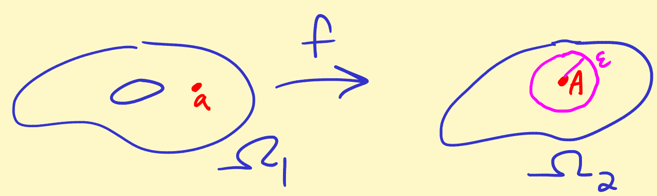
Ahlfors' pf: Poisson formula true when $a=0$. (It's just the ave prop then.) Hmmm. $\varphi_{-a}(z) = \frac{z+a}{1+\bar{a}z} = \varphi_a^{-1}(z)$. So $\varphi_{-a}(0)=a$

is analytic on $D_1(0)$ ↻ 1-1 onto and $\overline{D_1(0)}$ ↻ 1-1 onto.

Important fact: If $f:\Omega_1 \to \Omega_2$ is analytic and u is harmonic on Ω_2, the $u\circ f$ is harmonic on Ω_1.

Pf $\Delta(u\circ f) = $ mess ... use C-R equs ... $\equiv 0$.

Very easy pf:



f analytic $\Rightarrow$ f cont. So $\exists\, \delta>0$ such that

$D_\delta(a) \subset \Omega_1$ and $f(D_\delta(a)) \subset D_\varepsilon(A)$.

Let v be a harm conj for u on $D_\varepsilon(A)$.

$F = u+iv$ on $D_\varepsilon(A)$. $u = \mathrm{Re}\big[\underbrace{F\circ f}_{\text{analytic on } D_\delta(a)}\big]$ ✓

Big idea. Move a to origin where you can use ave prop.

$u(\varphi_{-a}(z))$ is harmonic

Value at $z=0$: $u\left(\frac{0+a}{1+\bar{a}\cdot 0}\right) = u(a)$

Ave prop: $$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\Big(\underbrace{\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}}_{e^{i\psi}}\Big)\, d\theta$$

Calculus: $$d\theta = \frac{1-|a|^2}{|a-e^{i\psi}|^2}\, d\psi$$ ← "just" calculus

Make change of variables. Get Poisson formula.

Important generalization Suppose u is continuous on $\overline{D_1(0)}$ and harmonic on $D_1(0)$. Then u is given by the Poisson formula in $D_1(0)$.

Pf: Let $u_r(z) = u(rz)$, $0<r<1$ const.

u_r is harmonic on $D_{1/r}(0)$, $\frac{1}{r}>1$. So Poisson formula holds for u_r:

$$\underbrace{u_r(a)}_{} = \int_0^{2\pi} \underbrace{P(a,\theta)}_{} \; \underbrace{u_r(e^{i\theta})}_{}\, d\theta$$

Let $r \nearrow 1$. $u(ra) \to u(a)$ as $r \nearrow 1$

nice and bounded and cont in θ when a fixed

$u(re^{i\theta}) \xrightarrow{\text{unif}} u(e^{i\theta})$ as $r \nearrow 1$ by unif continuity of u on $\overline{D_1(0)}$.

Exciting future: Poisson formula solves the Dirichlet problem!