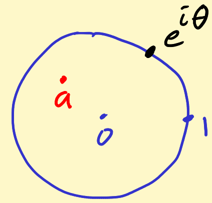
Lecture 35 The Dirichlet problem, Schwarz's theorem

Poisson kernel: $P(a,\theta) = \frac{1}{2\pi} \frac{1-|a|^2}{|a-e^{i\theta}|^2}$



Thm If u is continuous on $\overline{D_1(0)}$ and harmonic in $D_1(0)$, then

$$u(a) = \int_0^{2\pi} P(a,\theta)\, u(e^{i\theta})\, d\theta$$

Pf: Used trick: $u(rz) \xrightarrow{\text{unif}} u(z)$ as $r \nearrow 1$.

Remark Same trick works for Cauchy integral formula too.

Dirichlet problem Given continuous real valued fcn $U(\theta)$ on $[0,2\pi]$ with $U(0) = U(2\pi)$. [Ok to write $U(e^{i\theta})$ instead of $U(\theta)$.]

Is there a real valued fcn u such that

1) u is continuous on $\overline{D_1(0)}$,
2) u is harmonic on $D_1(0)$, and
3) $u(e^{i\theta}) = U(\theta)$ for $0 \le \theta \le 2\pi$?

Guess $u(a) = \int_0^{2\pi} P(a,\theta)\, U(\theta)\, d\theta$. Works! $\leftarrow$ Schwarz thm

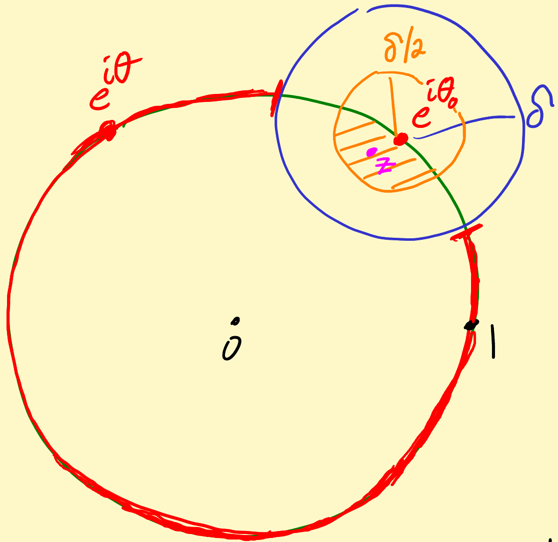
Solves heat prob: Temp on 2-D metal plate. Know temp on boundary. Let u = temp achieve "steady state". u solves D-prob. Why u is harmonic: Heat eqn: $c\Delta u = \frac{\partial u}{\partial t} \to 0$ steady state

4 key properties of $P(a,\theta)$

1) $P(a,\theta) > 0$ ✓ formula $(a \in D_1(0))$

2) $\displaystyle\int_0^{2\pi} P(a,\theta)\,d\theta = 1$ ✓ Poisson integral using $u \equiv 1$.

3)



$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

$P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$

$\longrightarrow 0$ uniformly on I_δ

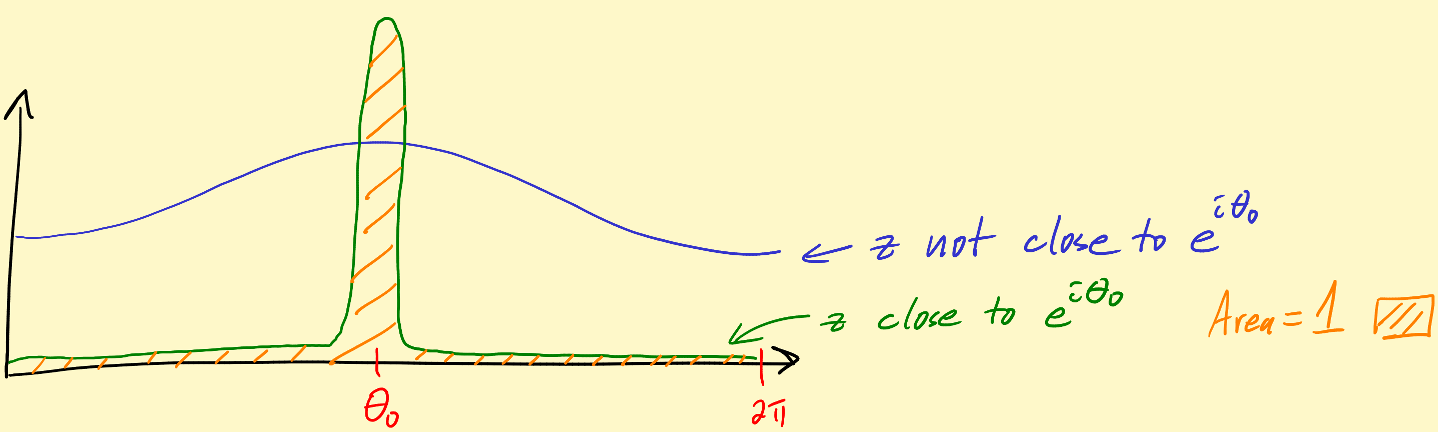
when $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$

and $z \to e^{i\theta_0}$.

(3) easy:
$$P(z,\theta) = \frac{1}{2\pi}\frac{1-|z|^2}{\underbrace{|z-e^{i\theta}|^2}_{|z-e^{i\theta}|>\frac{\delta}{2}}} \le \frac{1}{2\pi}\frac{(1-|z|)\overbrace{(1+|z|)}^{<2}}{\left(\frac{\delta}{2}\right)^2} < \frac{4}{\pi\delta^2}(1-|z|)\ ✓$$

4) $P(z,\theta)$ is harmonic in z on $D_1(0)$

Why $P(z,\theta) = \frac{1}{2\pi}\,\mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]$



<u>Pf</u> of Schwarz. Define $u(z) = \int_0^{2\pi} P(z,\theta)\, U(\theta)\, d\theta$

u solves D-prob.

$$= \frac{1}{2\pi} \mathrm{Re}\Big[\underbrace{\int_0^{2\pi} \Big[\frac{e^{i\theta}-z}{e^{i\theta}+z}\Big] U(\theta)\, d\theta}_{\text{analytic in } z \text{ via HWK 2, prob. 1}} \Big]$$

So u harmonic in $D_1(0)$.

Show u is continuous up to $C_1(0)$ and $\lim\limits_{z \to e^{i\theta_0}} u(z) = U(\theta_0)$.

$$u(z) - U(\theta_0) = \int_0^{2\pi} P(z,\theta) \big[U(\theta) - U(\theta_0) \big]\, d\theta$$

because $\int_0^{2\pi} P(z,\theta)\, d\theta = 1$

Let $\varepsilon > 0$. $\exists\ \delta > 0$ such that $|U(\theta) - U(\theta_0)| < \frac{\varepsilon}{2}$ when $e^{i\theta} \in D_\delta(e^{i\theta_0})$.

$$I = \underbrace{\int_{I_\delta} \underbrace{P(z,\theta)}_{\text{small}} \big[U(\theta) - U(\theta_0)\big]\, d\theta}_{I_1} + \underbrace{\int_{[0,2\pi] - I_\delta} P(z,\theta) \underbrace{\big[U(\theta) - U(\theta_0)\big]}_{|\ \ | < \frac{\varepsilon}{2}}\, d\theta}_{I_2}$$

Suppose $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$.

$$|I_1| \le \int_{I_\delta} |\text{~~~}|\, d\theta \le \frac{4}{\pi \delta^2} (1 - |z|)\, 2M\, \underbrace{\mathrm{Length}(I_\delta)}_{< 2\pi} \to 0 \text{ as } z \to e^{i\theta_0}.$$

where $M = \underset{[0,2\pi]}{\mathrm{Max}} |U|$. Can make $|I_1| < \frac{\varepsilon}{2}$ by taking z close enough to $e^{i\theta_0}$.

$$\text{Next } |I_2| \le \int_{[0,2\pi]-I_\delta} \underbrace{P(z,\theta)} \underbrace{|U(\theta) - U(\theta_0)|}_{<\frac{\varepsilon}{2}} d\theta$$

$|P(z,\theta)| = P(z,\theta)$ because $P>0$, #1.

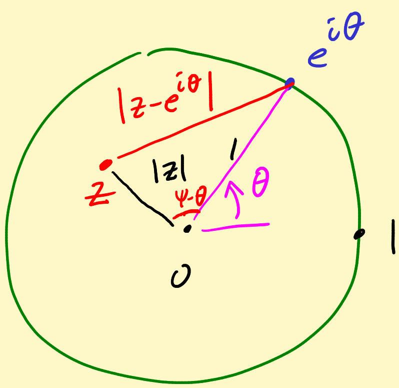
$$\le \frac{\varepsilon}{2} \int_{[0,2\pi]-I_\delta} P(z,\theta)\, d\theta < \frac{\varepsilon}{2} \underbrace{\int_0^{2\pi} P(z,\theta)\, d\theta}_{=1} < \frac{\varepsilon}{2}$$

(using $P>0$ again.)

Done! $|u(z) - U(\theta_0)| \le |I_1| + |I_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ ✓

Other formulas

$z = re^{i\psi}$



Law of cosines

$$u(re^{i\psi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\psi-\theta)}\, U(\theta)\, d\theta$$

$D_R(z_0)$ version: $f(z) = \dfrac{z-z_0}{R}$

$$u(z_0 + re^{i\psi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2-r^2}{R^2+r^2-2rR\cos(\psi-\theta)}\, u(z_0+Re^{i\theta})\, d\theta$$

Explosion of theorems!

<u>Monday</u> Continuous fcn with ave prop is harmonic!